AFCRL-64-290

ANTENNA LABORATORY

Technical Report No. 1

THE GENERALIZED SCATTERING MATRIX ANALYSIS OF WAVEGUIDE DISCONTINUITY PROBLEMS

by
JAMES R. PACE

Contract No. AF19(628)-3819

April 1964

Project 5635
Task 563502

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Department of Electrical Engineering
Engineering Experiment Station
University of Illinois
Urbana, Illinois

ABSTRACT

Three boundary value problems involving discontinuities in a parallel plate waveguide are solved. The three kinds of waveguide discontinuities studied are: 1) the metallic step discontinuity, 2) the inhomogeneous E-plane bifurcation, and 3) the trifurcation. The generalized scattering matrix technique, introduced by Mittra and Pace, is applied to solve these problems. The solutions obtained are formally exact and in series form. No restriction is made on the operating frequency of the waveguide. The solutions are equally valid for oversized waveguides, in which several modes are allowed to propagate down the guide at once. The results of numerical computations of the reflection coefficient for the dominant mode in the waveguide are reported. The computations were performed on a digital computer. Where possible, the author's results are compared with those published by Marcuvitz, Williams, and Cronson. Close agreement between the author's results and those of the above authors is noted.

ACKNOWLEDGEMENT

The author is indebted to Professor Raj Mittra, his advisor, and to Professor G. A. Deschamps for their advice and guidance. The author also wishes to thank Professor Y. T. Lo for reading the manuscript.

The work described in this report was sponsored in part by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, under contract AF 19(628)-3819, and in part by the Aeronautical Systems Division, Wright-Patterson Air Force Base, under contract AF 33(657)-10474.

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1. INTRODUCTION

Relatively few boundary value problems with applications in microwave or antenna engineering can be solved exactly. Broadly speaking, boundary value problems which are amenable to exact solution fall into one of two groups. With the first of these, the boundary conditions conform to one of the coordinate systems in which the scalar Helmholtz equation is separable. In this case, the partial differential equation is reduced to a set of ordinary differential equations, the solution of which is usually quite straightforward. A sizable volume of literature concerning the separation of variables technique is available. Morse and Feshbach[1] provides a comprehensive treatment of the topic.

There exists a second group of problems which can be solved exactly by means of integral transforms such as the Fourier and two-sided Laplace transforms. Often, problems of this sort may be formulated as an integral equation of the Wiener-Hopf type, or alternately in terms of certain special systems of infinite order linear algebraic equations. A Wiener-Hopf integral equation can be solved by the application of the Fourier transform and certain function-theoretic techniques. This method for solving a Wiener-Hopf integral equation is called the 'Wiener-Hopf technique'. Many papers on the application of the Wiener-Hopf technique are available in the literature. Noble[2] has published an excellent text concerned with both the theory and application of the technique. The exact solution of systems of infinite order linear algebraic equations by function-theoretic methods is discussed by Brillouin[3], Whitehead[4], Agronovich et. al.[5], Adonina et al.[6], and Hurd and Gruenberg[7].

Generally, however, the solution of a problem can only be formulated in terms of a differential or integral equation, or system of equations, which

can be solved only by approximate methods. Approximate methods are many and varied. They include variational and perturbational techniques, as well as finite-difference methods and the various iteration procedures commonly used to solve integral equations. Again, the volume of literature on the subject is enormous. Hartree[8] and Householder[9] have published well-known texts on numerical analysis. Goertzel and Tralli[10] is representative of the general references available on mathematical physics. A wide range of topics is covered, including chapters on perturbation of eigenvalues, variational estimates, etc.

A new technique for solving a class of boundary value problems is discussed in this thesis. It will be referred to hereafter as the generalized scattering matrix technique for reasons which will be made clear in the subsequent discussion. The application of this technique makes it possible to derive a formally exact solution, in series form, to problems for which only approximate solutions have been possible before.

It is believed that the generalized scattering matrix technique should have a broad range of applicability. The purpose of this thesis, however, is to demonstrate its usefulness for solving certain boundary value problems associated with discontinuities in a parallel plate waveguide. Specifically, three distinct problems are discussed. They are the E-plane metallic step discontinuity (Figure 1), the inhomogeneous E-plane bifurcation (Figure 2), and the trifurcated waveguide (Figure 3). The inhomogeneous E-plane bifurcation is an ordinary bifurcated waveguide modified by placing a dielectric in one of the smaller ducts of the waveguide, i.e., with reference to Figure 2, the dielectric is placed in region B.

Lewin[11], Collin[12], Ghose[13], Durrani[14], and Harvey[15] provide a survey of waveguide theory and existing techniques for solving waveguide discontinuity

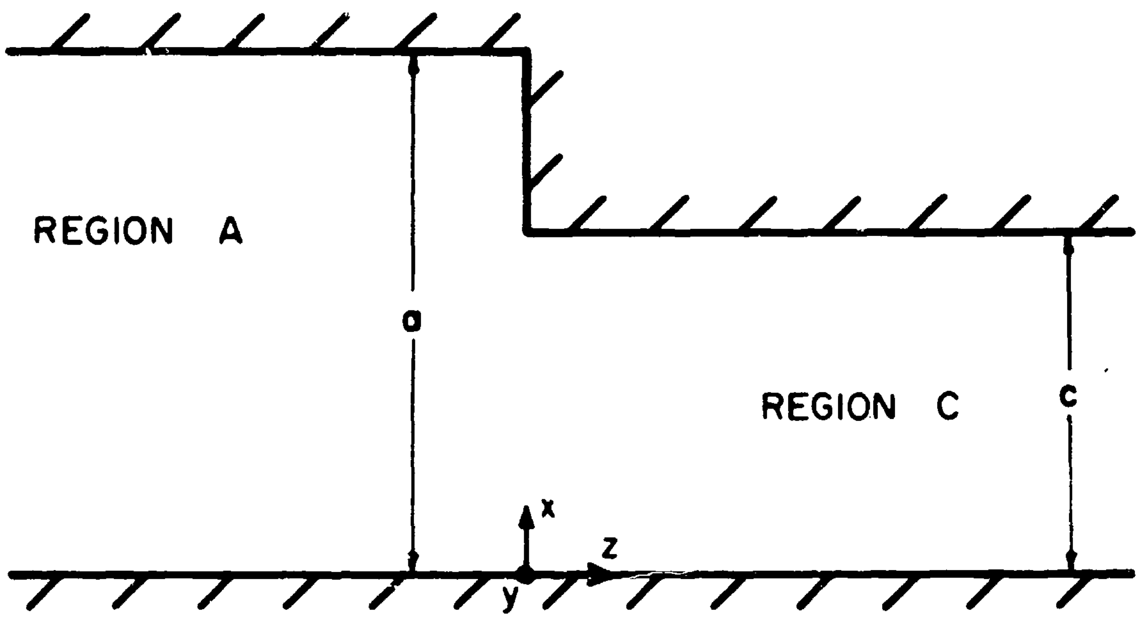

Figure 1. The E-plane metallic step discontinuity.

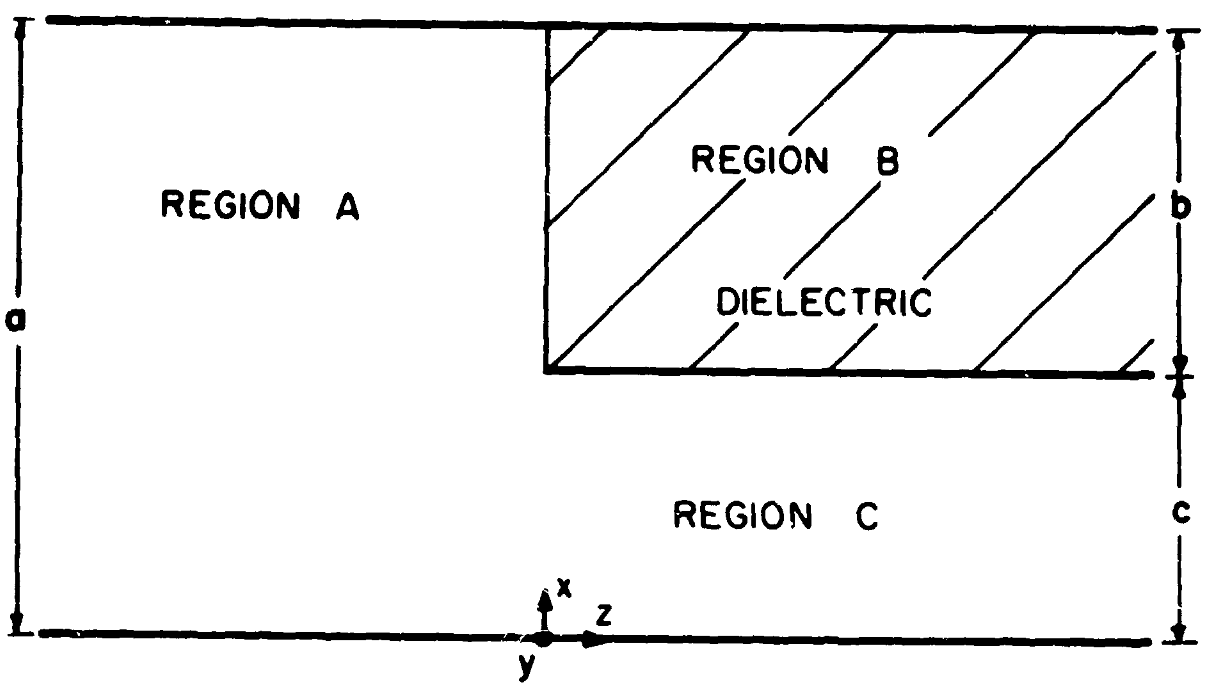

Figure 2. The inhomogeneous E-plane bifurcation.

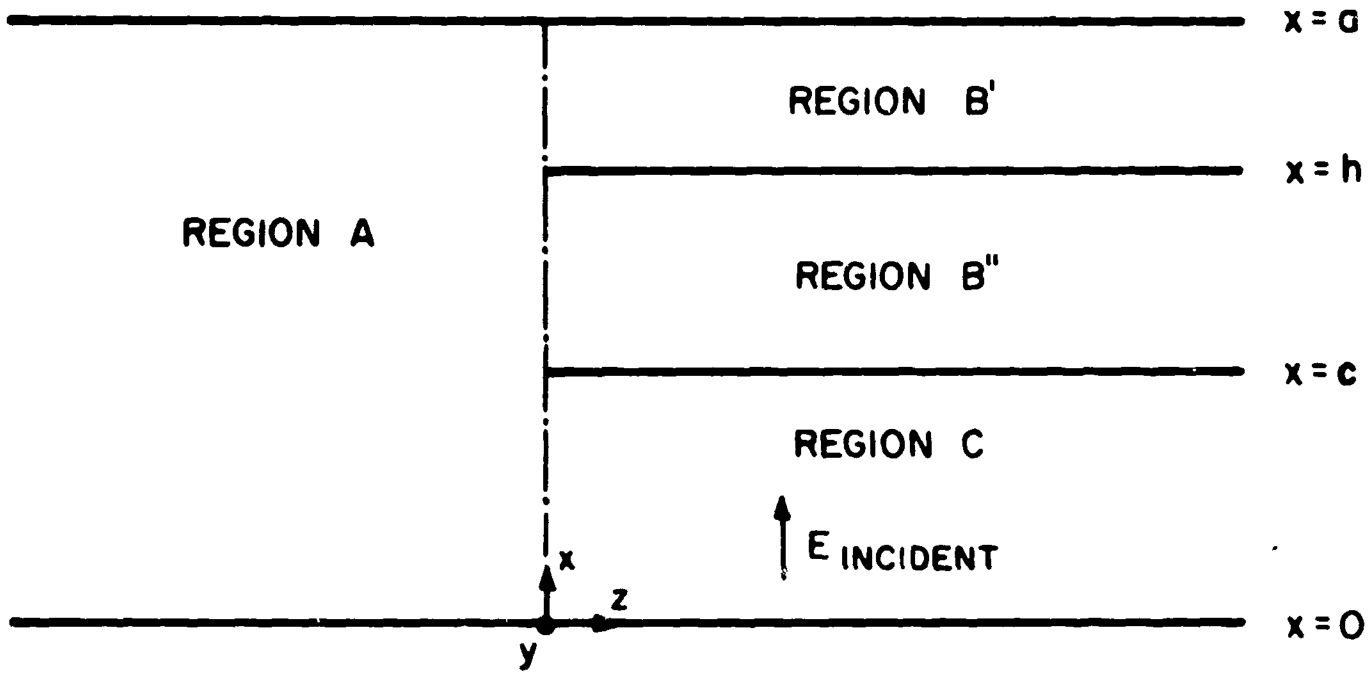

Figure 3. The trifurcated waveguide.

problems. Some papers of note specifically concerned with the three waveguide discontinuities discussed in this thesis have been published. They are discussed in Chapters 4, 5, and 6 in which the problems themselves are discussed.

This concludes the introduction. In the next chapter, the generalized scattering matrix technique is discussed.

2. THE GENERALIZED SCATTERING MATRIX TECHNIQUE

In this section of the paper, a new technique[16] for the solution of a class of boundary value problems arising in electromagnetic theory is presented. Although the technique should be applicable to other kinds of problems, too, the generalized scattering matrix technique is explained here by relating it to problems involving a class of waveguide discontinuities. In particular, the technique will be applied to three boundary value problems involving a parallel plate waveguide configuration. They are the inhomogeneous E-plane bifurcation, the metallic step discontinuity, and the trifurcated waveguide. Only brief mention of these problems is made in this section, however. The detailed solutions follow in later sections.

The key to the technique is the identification of an auxiliary problem associated with the particular problem to be solved. Each of the three problems discussed in detail in this thesis has a common auxiliary problem. This is the boundary value problem associated with a semi-infinite bifurcation in a parallel plate waveguide (refer to Figure 4). The geometry of the auxiliary problem is such that it can be modified in a straightforward manner so as to be made identical with the geometry of the original problem. For instance, if region B of the bifurcation is filled with dielectric, then the resulting configuration is just that of the inhomogeneous E-plane bifurcated waveguide. The dielectric can be regarded as a termination or load placed in region B. Likewise, the appropriate modification in the case of the metallic step discontinuity is made by placing a perfectly reflecting wall in region B flush with the plane of the junction ($z = 0$). In the case of the trifurcation, the auxiliary problem is modified by placing a second bifurcation in region B.

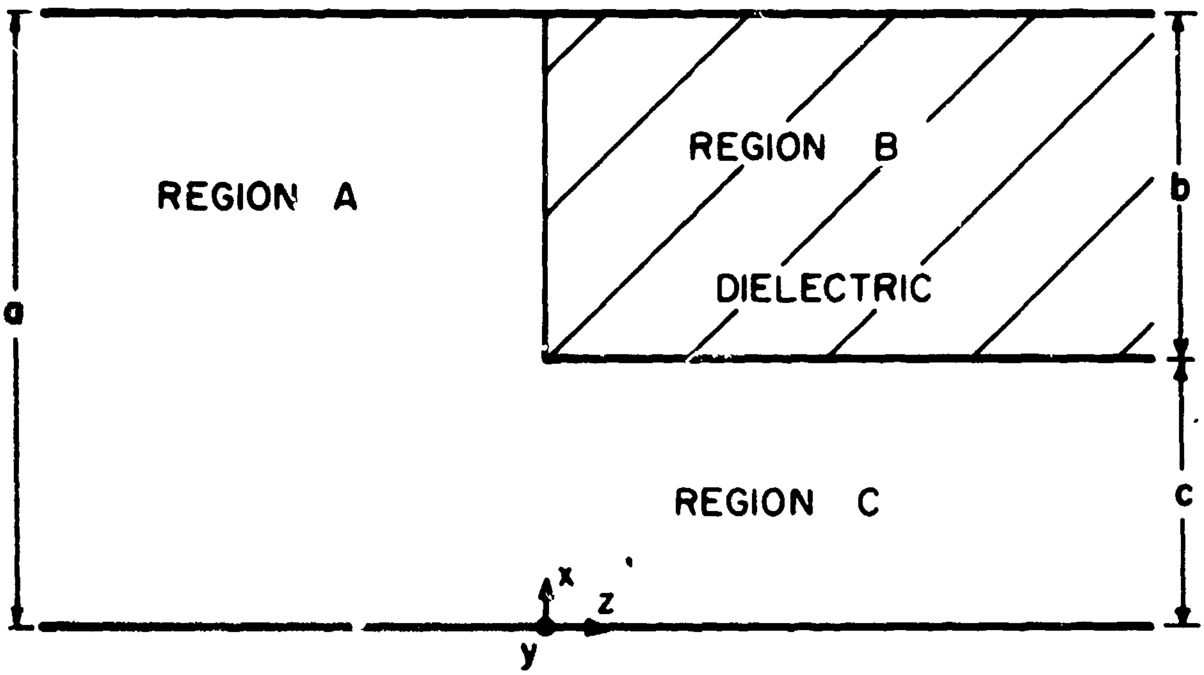

Figure 4. The inhomogeneous E-plane bifurcation.

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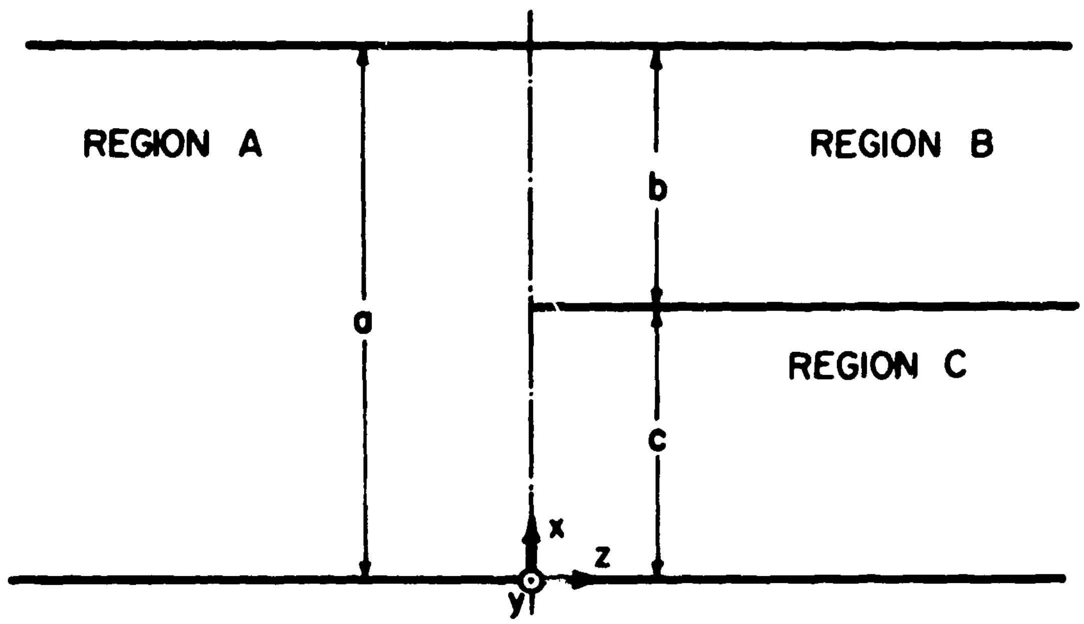

The auxiliary problem.

(Use this Figure 4 in place of the page 8 Figure 4 in text)

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The auxiliary problem is characterized by the fact that it can be solved exactly. The problem of the semi-infinite bifurcation can be solved exactly by any one of several methods. For instance, it can be handled by the Wiener-Hopf[17] or function-theoretic technique[7], or the direct solution of an infinite set of linear algebraic equations[18].

The way in which the auxiliary problem is used to solve the three above mentioned problems will be discussed next.

2.1 Development of the Series Expansion

It is common practice to think of the bifurcated waveguide in terms of a transmission line analogy. If only the dominant mode of the guide is allowed to propagate, as usually is the case, the bifurcated waveguide is regarded as a 3-port network. One port of the network is associated with the propagating mode in each of the three regions A, B, and C. A port may be regarded as the terminals of a transmission line. If the problem of multi-mode propagation is under consideration, then the network is regarded as an N-port system, N being the total of the propagating modes in the three regions.

In the neighborhood of the edge of the discontinuity, an infinite number of evanescent modes are excited by the diffraction of an incident plane wave. The effect of these evanescent modes can be represented in terms of lumped reactances in the equivalent circuit as these modes in physical terms represent stored energy. Of course, the numerical values of these lumped reactances must be determined by solving the boundary value problem.

The network of lumped reactances and transmission lines can be concisely described in mathematical terms by means of an impedance, admittance, or scattering matrix. The order of any of these matrices will be N where N is the total number of propagating modes in the three regions A, B, and C.

In the technique introduced in the thesis, the bifurcated guide is regarded as a 3N-port network, N being infinitely large. A port of the network is assigned to each of the propagating and evanescent modes. If the ports of the network are terminated with the proper set of reflectances, a network corresponding to either the step discontinuity, the inhomogeneous bifurcated waveguide, or the trifurcation is achieved.

The concept of a scattering matrix of infinite order is introduced. While the mechanism of its application is conventional, it differs from the scattering matrices ordinarily defined in the literature. For one thing, the concept of the scattering coefficient is extended to cover evanescent modes. The following discussion is concerned with the derivation of the generalized scattering matrix of infinite order as applied to waveguide discontinuity problems of the kind discussed in this thesis.

With reference to Figure 4, let $S^{\alpha\alpha}$, $\alpha = A$, B, or C represent the self-scattering matrices of the auxiliary problem. Let $S^{\alpha\beta}$, $\alpha = A$, B, or C and $\beta = A$, B, or C but $\alpha \neq \beta$, be the mutual-scattering matrices.

The interpretation of the scattering matrices is as follows. Consider that regions B and C are terminated in reflectionless loads. Then, if the n^{th} transverse magnetic mode is incident in region A, fields will be reflected in region A and transmitted to regions B and C. These fields can be determined by solving the auxiliary problem. The resultant electromagnetic fields are expressible entirely in terms of transverse magnetic modes. Thus, the total electric field is expressible in terms of the total H_y field. In turn, H_y can be written in terms of eigenfunction expansions with constant coefficients appropriate to regions A, B and C. In this thesis, the H_y component of the transverse magnetic field is expanded in each of the three

regions A, B, and C in terms of Fourier cosine series of the form $\sum_n d_n \phi_n$ where in region A, $\phi_n = \cos \frac{\pi n x}{a}$, in region B, $\phi_n = \cos \frac{\pi n (x-a)}{b}$, and in region C, $\phi_n = \cos \frac{\pi n x}{c}$. The mode coefficient of the m^{th} mode referred to the plane of the junction $(z = 0)$, appearing in the expansion of H_y will be defined as the 'amplitude' of the mode. If the amplitude of the n^{th} mode incident in region A is one, or in other words, the incident H_y field is given by $\cos \frac{\pi n x}{a}$, the total H_y field reflected in region A at $z = 0$ is given by $-\sum_{m=0}^{\infty} S_{mn}^{AA} \cos \frac{\pi m x}{a}$. The total H_y field transmitted say to region B at $z = 0$ is given by $\sum_{m=0}^{\infty} S_{mn}^{BA} \cos \frac{\pi m (x-a)}{b}$. Thus, if the amplitude of the n^{th} mode incident in region A is one, the amplitude of the m^{th} mode scattered in region A will be $-S_{mn}^{AA}$ and the amplitude of the m^{th} mode transmitted to region B will be S_{mn}^{BA}. S_{mn}^{AA} and S_{mn}^{BA} are the general matrix elements of S^{AA} and S^{BA}, respectively. The other matrices are defined in a similar manner.

Note the scattering coefficients are defined in this thesis in a manner consistent with the sign convention followed in electromagnetic theory when defining the reflection and transmission coefficients in terms of the voltage or transverse electric field rather than in terms of the current or transverse magnetic field. Since the coefficients of the eigenfunction expansion of H_y are used directly to define the various scattering coefficients, care must be taken to assign the proper sign to the ratio of the amplitude of the scattered mode to the amplitude of the incident mode if the definitions are to be consistent with this sign convention. Thus, the amplitude of the m^{th} mode comprising H_y in region A is given by $-S_{mn}^{AA}$, and not by just S_{mn}^{AA}.

It should be noted that ordinarily the mode amplitudes are normalized so that a propagating mode carries unit power. However, since the scattering matrix has been generalized to include evanescent modes, it is inappropriate

to normalize the mode amplitudes in the usual manner. In this thesis, the scattering coefficients are defined as the ratio of the amplitude of a scattered mode to the amplitude of the incident mode which is taken to be unity. One consequence of this definition, however, is that the various scattering matrices are non-symmetric.

Suppose now that region B is modified by placing an obstacle in it such as a perfectly reflecting wall. In terms of the N-port network representation, the load in region B can be represented by a reflectance matrix Γ^B and a transmission matrix Φ^B. The meaning of Γ^B and Φ^B can be interpreted as follows. Let $\bar{t}$ be a vector associated with the field distribution of a wave progressing toward positive z in region B. The elements of the vector $\bar{t}$, of course, are the amplitudes of the modes of the eigenfunction expansion of the arbitrary field in region B. The amplitudes are referred to the plane z = 0. If this is so, then $\Gamma^B \bar{t}$ gives the reflected field and $\Phi^B \bar{t}$ gives the transmitted field, both referred to the plane of the junction at z = 0. $\Gamma^B \bar{t}$ is progressing in region B toward negative z away from the load. The elements of $\Gamma^B \bar{t}$ are the amplitudes of the modes in the reflected field. Similarly, $\Phi^B \bar{t}$ is progressing in region B toward positive z and the elements of $\Phi^B \bar{t}$ are the amplitudes of the modes comprising the transmitted field. As an example, if a perfectly reflecting wall in region B is flush with the plane of the junction, $\Gamma^B = -I$ where I is the identity matrix and $\Phi^B = 0$.

With Γ^B and Φ^B defined, consider the situation shown in Figure 5. The usual boundary value problem associated with this kind of geometry involves a monochromatic signal excited in some region of the guide and propagating toward the junction. One is interested in determining the fields diffracted by the junction, or at least some part of them. For example, the reflection

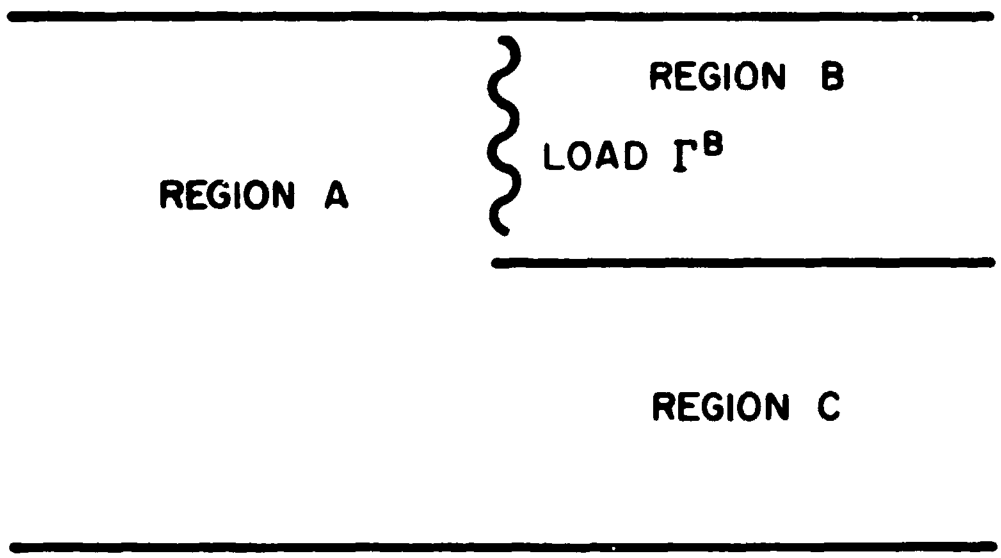


Figure 5. Auxiliary problem modified by a load placed in region B.

and transmission coefficients for the dominant mode are often sought. One way to express the desired fields is in terms of their eigenfunction expansions with constant coefficients. The generalized scattering matrix technique enables one to write the coefficients of the eigenfunction expansion in terms of the scattering coefficients of the auxiliary problem and the load in region B. The derivation of the relationship between the eigenfunction coefficients and the scattering matrices follows.

Suppose region A is excited by a TEM mode. Suppose also that region B is terminated by a load characterized by the reflectance matrix Γ^B. The TEM mode will be scattered and reflected first by the bifurcation in the waveguide. The field reflected into region A can be characterized by the vector $\bar{r}_o$ where $\bar{r}_o = S^{AA}\,\bar{a}$. By definition $\bar{a} = (1, 0, 0, \ldots)^T$ where T means the transpose. (Actually, in this particular case $\bar{r}_o = 0$). Furthermore, a wave will be transmitted to region B. Let $\bar{t}_o$ characterize this wave where $\bar{t}_o = S^{BA}\,\bar{a}$. The wave will be reflected by the load in region B. The reflected wave is characterized by $\Gamma^B\bar{t}_o$. The reflected wave progresses toward the junction in the negative z direction where it is diffracted by the edge of the bifurcation. A field $\bar{r}_1 = S^{AB}\Gamma^B\bar{t}_o$ is transmitted to region A and a field $\bar{t}_1 = S^{BB}\Gamma^B\bar{t}_o$ is reflected in region B. This field will also be reflected back by the termination and this process of multiple reflection will be continued. All of the contributions in region A due to this process can be written symbolically as

$$\bar{S}_{in} = \sum_{n=0}^{\infty} \bar{r}_n = S^{AA}\,\bar{a} + S^{AB}\Gamma^B\bar{t}_o + S^{AB}\Gamma^B\; S^{BB}\Gamma^B\bar{t}_o + \ldots \qquad (1)$$

This is recognized as a Neumann type series. The Neumann series can be summed in the usual manner, and Equation (1) can be written as

$$\bar{S}_{in} = S^{AA}\,\bar{a} + S^{AB}\,\Gamma^B\,(I - S^{BB}\,\Gamma^B)^{-1}\,S^{BA}\bar{a} \qquad (2)$$

$\bar{S}_{in}$ is given by $(R_A, -A_1, -A_2, \ldots)^T$ where R_A is the voltage reflection coefficient for the dominant mode and the A_n's are the coefficients of the Fourier series expansion of H_y in region A. Note that the difference in the signs between R_A and the A_n's above is due to the aforementioned definitions of the scattering coefficients. The convergence of the Neumann series is discussed in Section 2.2.

The discussion of the derivation of Equation (2) in the preceding paragraph depends on an intuitive understanding of the physical processes involved in the chain of multiple reflections. In order to provide a lucid explanation, it was tacitly assumed that the load in region B was recessed some arbitrary distance δ into region B and away from the edge of the bifurcation. Refer to Figure 6. If indeed this is the case, then the proof of the convergence of Equation (1) is very much simplified since all of the higher order evanescent modes would be damped out quite rapidly. However, this crutch is not necessary. In the limit, δ can be zero and it will be shown that the series even then is convergent. Indeed, convergence is proven without reference to the specific expressions for the various scattering coefficients.

Using reasoning similar to that employed in deriving Equations (1) and (2), an expression for the fields in say region C can be written as

$$\bar{S}_{CA} = S^{CA}\,\bar{a} + S^{CB}\,\Gamma^B\,(I - S^{BB}\,\Gamma^B)^{-1}\,S^{BA}\,\bar{a} \qquad (3)$$

where $\bar{S}_{CA} = (T_{CA}, C_1, C_2, \ldots)^T$. T_{CA} is the transmission coefficient for the dominant mode from region A to region C and the C_n's are the higher order coefficients of the Fourier series expansion for the solution field in region C.

In a similar manner, the mode coefficients of the eigenfunction expansion of H_y in region B can be written as

$$\bar{S}_{BA} = \Phi^B\,(I - S^{BB}\,\Gamma^B)^{-1}\,S^{BA}\,\bar{a} \qquad (4)$$

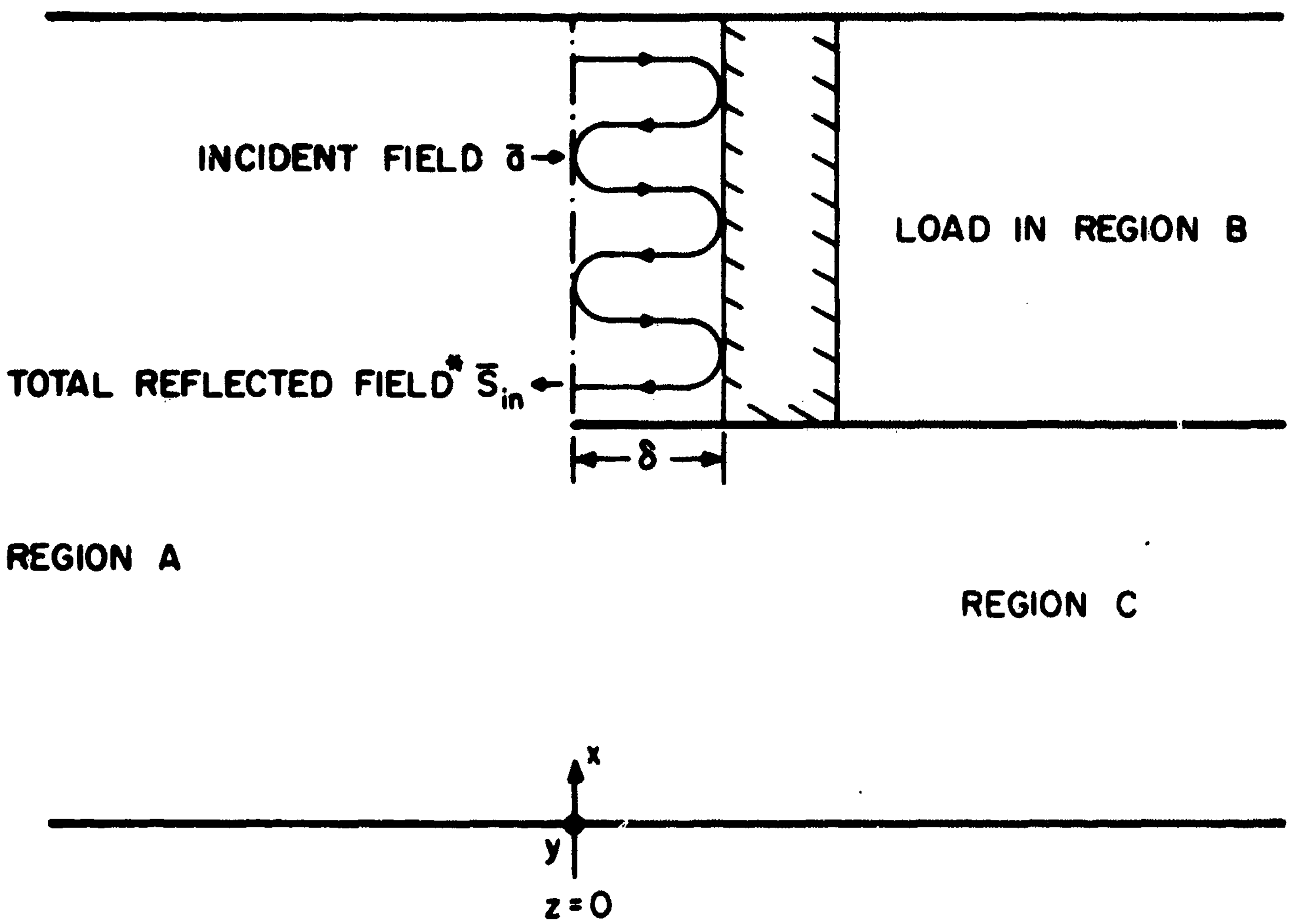


Figure 6. Multiple scattering by load in region B.

* $\bar{S}_{in} = S^{AA}\bar{a} + S^{AB}\,\Gamma^{B}\,(I - S^{BB}\,\Gamma^{B})^{-1}\,S^{BA}\bar{a}$

where $\overline{S}_{BA} = (T_{BA}, B_1, B_2, \ldots)^T$. T_{BA} is the transmission coefficient for the dominant mode from region A to region B and the B_n's are the higher order mode coefficients of the Fourier series expansion for H_y in region B.

Equations (2), (3), and (4) apply to the specific case of a signal exciting region A and a load placed in region B. However, similar equations can easily be derived for other situations. Also, note that loads can be placed in two regions in the same problem. For instance, the problem of the capacitive diaphragm in a waveguide can be solved through the device of placing a magnetic wall in region C and an electric wall in region B. The method of multiple scattering has been applied to this problem[19] for the case of a semi-septum. The solution was shown to be quite straightforward.

2.2 The Proof of the Convergence of the Neumann Series Expansion

In this section, the convergence of the Neumann series expansion given by

$$(I - S^{BB}\,\Gamma^B)^{-1} = I + S^{BB}\,\Gamma^B + S^{BB}\,\Gamma^B S^{BB}\,\Gamma^B + \ldots \tag{5}$$

is proven. First, the convergence of the series

$$(I - S^{BB})^{-1} = I + S^{BB} + S^{BB} S^{BB} + \ldots \tag{6}$$

is demonstrated. Next, it is demonstrated that if series (6) is convergent, series (5) is also convergent.

Let a_n and b_n be the amplitudes of the n^{th} mode incident and reflected in region B, defined at the plane of the junction ($z = 0$). Furthermore, let e_n and i_n be the voltage and current for the n^{th} mode in region B, defined at the plane of the junction ($z = 0$). For propagating modes, and $b_n = 0$, $\frac{1}{2} e_n i_n^* = \frac{1}{2} Z_n^{(p)} |a_n|^2$, where $Z_n^{(p)} = \frac{b}{2\omega\epsilon_o} \sqrt{k^2 - \left(\frac{\pi n}{b}\right)^2}$. Similarly, for evanescent modes, also with $b_n = 0$, one has $\frac{1}{2} e_n i_n^* = -\frac{j}{2} Z_n^{(e)} |a_n|^2$, where $Z_n^{(e)} = \frac{b}{2\omega\epsilon_o} \sqrt{\left(\frac{\pi n}{b}\right)^2 - k^2}$.

For propagating modes, e_n and i_n can, in general, be expressed in terms of a_n and b_n as

$$e_n = Z_n^{(p)} (a_n + b_n) \tag{7}$$

and

$$i_n = (a_n - b_n) \tag{8}$$

For evanescent modes, e_n and i_n can be expressed as

$$e_n = -jZ_n^{(e)} (a_n + b_n) \tag{9}$$

and

$$i_n = (a_n - b_n) \tag{10}$$

With the reference plane chosen in region B at the plane $z = 0$, regions A and C can be grouped together and regarded as a termination. A relationship between the terminal voltages and currents, the stored energy, and the power delivered to the termination is derived by Montgomery, et al.[20]

$$\frac{1}{2} \sum_{n=0}^{\infty} e_n i_n^* = 2j\omega (W_H - W_E) + P \tag{11}$$

where W_H = average magnetic energy stored in the termination

W_E = average electric energy stored in the termination

P = average power delivered to the termination.

To simplify the following discussion, assume that only the TEM mode propagates in region B. Substituting (7), (8), (9) and (10) into Equation (11), one obtains

$$Z_o^{(p)} (a_o + b_o) (a_o^* - b_o^*) - j\sum_{n=1}^{\infty} Z_n^{(e)} (a_n + b_n) (a_n^* - b_n^*)$$
$$= 4j\omega (W_H - W_E) + 2P \tag{12}$$

Equating the real parts of Equation (12), one obtains

$$Z_o^{(p)} \left(|a_o|^2 - |b_o|^2 \right) - j \sum_{n=1}^{\infty} Z_n^{(e)} (a_n^* b_n - a_n b_n^*) = 2P \qquad (13)$$

Equating the imaginary parts of Equation (12), one derives the equation

$$Z_o^{(p)} (a_o b_o^* - a_o^* b_o) + j \sum_{n=1}^{\infty} Z_n^{(e)} \left(|a_n|^2 - |b_n|^2 \right) = 4j\,(W_E - W_H) \quad (14)$$

Now let $\bar{b} = S^{BB}\,\bar{a}$ where $\bar{a}$ is an eigenvector of S^{BB}, so that if λ is an eigenvalue

$$\bar{b} = \lambda\,\bar{a} \qquad (15)$$

and

$$b_n = \lambda\,a_n \qquad (16)$$

Substituting Equation (16) into Equations (13) and (14), one obtains

$$Z_o^{(p)} |a_o|^2 \left(1 - |\lambda|^2\right) + j \sum_{n=1}^{\infty} (\lambda^* - \lambda)\, Z_n^{(e)} |a_n|^2 = 2P \qquad (17)$$

and

$$Z_o^{(p)} |a_o|^2 (\lambda^* - \lambda) + j \sum_{n=1}^{\infty} \left(1 - |\lambda|^2\right) Z_n^{(e)} |a_n|^2 = 4\,j\omega(W_E - W_H) \quad (18)$$

Equations (17) and (18) can be treated as two equations in the two unknowns $(\lambda^* - \lambda)$ and $(1 - |\lambda|^2)$. Solving for $(1 - |\lambda|^2)$, one gets

$$1 - |\lambda|^2 = \frac{2P\, Z_o^{(p)} |a_o|^2 + 4\omega(W_E - W_H) \sum_{n=1}^{\infty} Z_n^{(e)} |a_n|^2}{\left(Z_o^{(p)} |a_o|^2\right)^2 + \left(\sum_{n=1}^{\infty} Z_n^{(p)} |a_n|^2\right)^2} \qquad (19)$$

Note that $Z_o^{(p)}$ and $Z_n^{(e)}$ are positive real quantities and $W_E > W_H$ for the case of TM modes. Thus,

$$|\lambda| < 1 \qquad (20)$$

A necessary and sufficient condition for the convergence of the Neumann series $I + M + M^2 + \ldots$ is that the eigenvalues of M satisfy the inequality $|\lambda| < 1$. Thus, the convergence of $(I + S^{BB})$ is proved.

The proof of the convergence of $I + S^{BB} + S^{BB}S^{BB} \ldots$ for the case of TE modes is formally the same as the proof given above for TM modes. In this case $W_H > W_E$, but the voltage e_n for the n^{th} non-propagating voltage is normalized differently. In this case, $e_n = j Z_n^{(e)} (a_n + b_n)$, $Z_n^{(e)} > 0$.

There is an alternate condition, necessary and sufficient, for the convergence of a Neumann series (see, for instance, Friedman[21]). If $\bar{a}'$ is an arbitrary vector and $\bar{b}' = M\bar{a}'$, then the Neumann series $I + M + M^2 + \ldots$ converges absolutely if $|\bar{b}'| < |\bar{a}'|$ where $|\bar{a}'|$ is finite. Thus, from the proof of the convergence of $I + S^{BB} + S^{BB} S^{BB} + \ldots$, one already knows that if $\bar{b}' = S^{BB} \bar{a}'$, then $|\bar{b}'| < |\bar{a}'|$. Now, let $\bar{a}' = \Gamma^B \bar{a}''$. Then, if $\bar{b}'' = S^{BB}\Gamma^B\bar{a}''$, $|\bar{b}''| < |\Gamma^B \bar{a}''|$.

Now, one can follow the same argument for Γ^B as was used for S^{BB} to derive Equation (19), again. In this case, however, it is possible that $P = 0$ and $W_E = W_H$ as would be the case if $\Gamma^B = \pm I$. Hence, if η is any eigenvalue of Γ^B, $|\eta| \leq 1$, and it follows that $|\Gamma^B \bar{a}''| \leq |\bar{a}''|$. Thus, $|\bar{b}''| < |a|$, showing that the Neumann series $I + S^{BB}\Gamma^B + S^{BB}\Gamma^B S^{BB}\Gamma^B + \ldots$ is absolutely convergent and can be summed to $(I - S^{BB}\Gamma^B)^{-1}$. The proof is essentially the same if the case of several modes propagating is considered.

In Chapters 4, 5, and 6, $(I - S^{BB}\Gamma^B)$ is truncated and then inverted. The Neumann series expansion was used in computing the inverse. For all cases considered, it was found that the Neumann series can be truncated after 20 to 30 terms. The computation of the inverse of $(I - S^{BB}\Gamma^B)$ using the Neumann series expansion is straightforward even when the rank of the truncated matrix

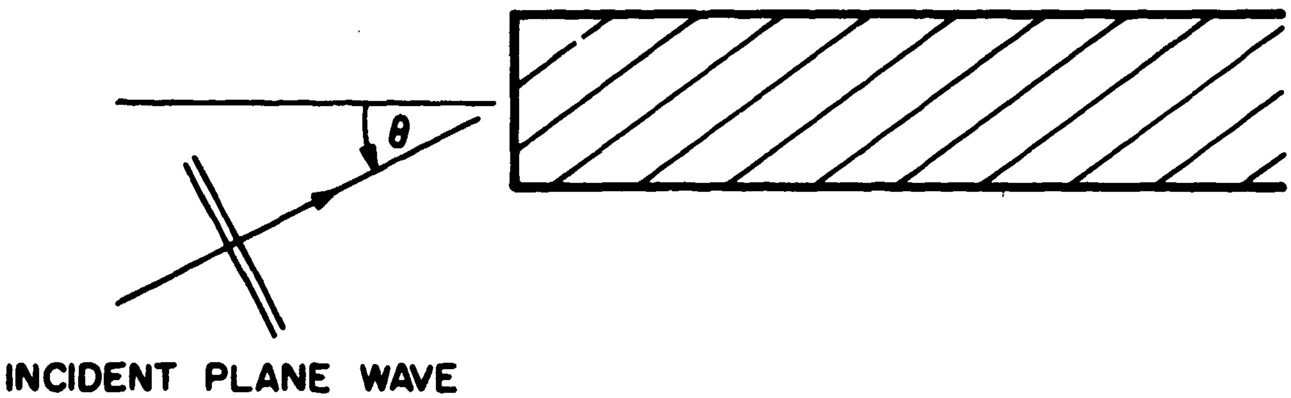
is relatively large. Of course, if the rank of the truncated matrix $(I - S^{BB}\Gamma^{B})$ is sufficiently small, then the matrix can be inverted directly.

2.3 General Comments on the Technique

The introduction of an auxiliary problem of the kind discussed above as an aid to the solution of certain kinds of boundary value problems is rather new. A search of the literature has uncovered only one paper in which the author uses a similar device. W. E. Williams[22] uses the Laplace transform in the formulation of the step discontinuity problem. He applies the Wiener-Hopf technique and derives an associated set of infinite order linear algebraic equations. The auxiliary problem is introduced by Williams as a preliminary step to solving the infinite set of equations. This is to be contrasted with the use of the auxiliary problem as discussed in this thesis. Using the generalized scattering matrix technique, the solution to the problem is expressed in terms of the scattering coefficients of the auxiliary problem as a rapidly convergent series.

Also, the applicability of the generalized scattering matrix technique should be rather broad. The method is not necessarily restricted to waveguide problems. For example, it is suggested that the problem of the diffraction of a plane wave by a thick, conducting half-plane may be solved by means of the new technique. The auxiliary problem suggested is the boundary value problem associated with a pair of parallel, semi-infinite plates in free space (Figure 7).

A brief list of problems suggested for future study is given in Chapter 7 of the thesis.



(a)

Figure 7a. Diffraction of plane wave by a thick half-plane.

(b)

Figure 7b. The proposed auxiliary problem: parallel-plate waveguide in space.

3. DERIVATION OF THE SCATTERING COEFFICIENTS

The elements of the self-scattering matrices $S^{\alpha\alpha}$, $\alpha = A$, B, or C, and the mutual-scattering matrices $S^{\alpha\beta}$, $\alpha = A$, B, or C and $\beta = A$, B, or C but $\alpha \neq \beta$, are determined by solving the boundary value problem associated with a semi-infinite bifurcation in a parallel plate waveguide. The problem must be solved for an arbitrary TM_{no} mode incident from one of the three regions A, B, or C.

It may be shown for the problem under consideration that the only modes excited by the discontinuity with an arbitrary TM_{no} mode incident are the TM_{no} modes. The non-vanishing field components can be derived from a single scalar function $\phi(x, z)$ which is identical to the y-component of the H-field. Using the coordinate system illustrated in Figure 4 the three components of the field can be written as

$$H_y = \phi \tag{21}$$

$$E_x = \frac{1}{j\omega\epsilon_o} \frac{\partial\phi}{\partial z} \tag{22}$$

and

$$E_z = \frac{-1}{j\omega\epsilon_o} \frac{\partial\phi}{\partial x} \tag{23}$$

A harmonic time variation of the type $e^{j\omega t}$ is assumed throughout. The scalar function $\phi(x, z)$ must satisfy the two-dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \phi = 0, \quad k = \frac{2\pi}{\lambda} \tag{24}$$

together with

$$\frac{\partial\phi}{\partial x} = 0, \; x = 0, a, \text{ all } z \text{ and } x = c, \; z > 0 \tag{25}$$

and the edge condition

$$\left| \nabla \phi \right| = O(d^{-1/2}), \; d \longrightarrow o \tag{26}$$

where $d = \left[z^2 + (x - c)^2 \right]^{1/2}$

Initially, it is assumed that the waveguide is excited from each of the three regions A, B, and C by an arbitrary transverse magnetic mode of order ℓ, q, and r, respectively. It is assumed that the waveguide dimensions are such that each of the incident modes is a propagating mode. Thus, in general, there will be several propagating modes in each of the three regions of the waveguide. Each of the propagating modes must satisfy the radiation condition at infinity.

3.1 Derivation of the Infinite Sets of Equations

In each of the three regions labeled A, B, and C the function $\phi(x,z)$ can be written in terms of the appropriate eigenfunction expansion, in this case a cosine series. Thus, in region A

$$\phi_A = A \cos\left(\frac{\pi \ell x}{a}\right) e^{-j\alpha_\ell z} + \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n x}{a}\right) e^{j\alpha_n z} \tag{27}$$

where

$$\alpha_n = \sqrt{k^2 - \left(\frac{\pi n}{a}\right)^2}, \quad k > \frac{\pi n}{a}$$

$$= -j\sqrt{\left(\frac{\pi n}{a}\right)^2 - k^2}, \quad \frac{\pi n}{a} > k$$

In region B,

$$\phi_B = B \cos\left(\frac{\pi q (x-a)}{b}\right) e^{j\beta_q z} + \sum_{n=0}^{\infty} B_n \cos\left(\frac{\pi n (x-a)}{b}\right) e^{-j\beta_n z} \tag{28}$$

where

$$\beta_n = \sqrt{k^2 - \left(\frac{\pi n}{b}\right)^2}, \quad k > \frac{\pi n}{b}$$

$$= -j\sqrt{\left(\frac{\pi n}{b}\right)^2 - k^2}, \quad \frac{\pi n}{b} > k$$

In region C,

$$\phi_C = C \cos\left(\frac{\pi r x}{c}\right) \epsilon^{j\gamma_r z} + \sum_{n=0}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right) \epsilon^{-j\gamma_n z} \tag{29}$$

where

$$\gamma_n = \sqrt{k^2 - \left(\frac{\pi n}{c}\right)^2}, \quad k > \frac{\pi n}{c}$$

$$= -j\sqrt{\left(\frac{\pi n}{c}\right)^2 - k^2}, \quad \frac{\pi n}{c} > k$$

A, B, and C are the amplitudes of the modes incident in regions A, B, and C, respectively. The integers ℓ, q, and r are arbitrary.

The coefficients A_n, B_n, and C_n are related through the requirement that the transverse E-field and H-field must be continuous across the plane of the discontinuity (z = 0). Matching the transverse E-field and H-field across the boundary yields four sets of equations:

$$A \cos\left(\frac{\pi \ell x}{a}\right) + \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n x}{a}\right) = B \cos\left(\frac{\pi q(x-a)}{b}\right) + \sum_{n=0}^{\infty} B_n \cos \frac{\pi n(x-a)}{b} \tag{30}$$

and

$$\alpha_\ell A \cos\left(\frac{\pi \ell x}{a}\right) - \sum_{n=0}^{\infty} \alpha_n A_n \cos\left(\frac{\pi n x}{a}\right) = -\beta_q B \cos\left(\frac{\pi q(x-a)}{b}\right) + \sum_{n=0}^{\infty} \beta_n B_n \cos\left(\frac{\pi n(x-a)}{b}\right) \tag{31}$$

valid in the interval $c \leq x \leq a$

and also

$$A \cos\left(\frac{\pi \ell x}{a}\right) + \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n x}{a}\right) = C \cos\left(\frac{\pi r x}{c}\right) + \sum_{n=0}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right) \tag{32}$$

and

$$\alpha_\ell A \cos\left(\frac{\pi \ell x}{a}\right) - \sum_{n=0}^{\infty} \alpha_n A_n \cos\left(\frac{\pi n x}{a}\right) = -\gamma_r C \cos\left(\frac{\pi r x}{c}\right) + \sum_{n=0}^{\infty} \gamma_n C_n \cos\left(\frac{\pi n x}{c}\right) \tag{33}$$

which are valid in the interval $0 \leq x \leq c$.

For the present, it shall be assumed that a/b is not an integer.* If both sides of Equations (30) and (31) are multiplied by $\cos \frac{\pi s(x-a)}{b}$, where s is an arbitrary positive integer, and both sides of the equations are then integrated between the limits c to a, two sets of infinite equations can be derived. These are:

$$b(A\,\delta^o_\ell + A_o)\delta^o_s + (-1)^s \frac{\pi}{a} \frac{A'}{\alpha_\ell^2 - \beta_s^2}(1 - \delta^o_\ell) - (-1)^s \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n^2 - \beta_s^2}$$

$$= bB_s\delta^o_s + \frac{b}{2} B_s\,(1-\delta^o_s) + bB\delta^o_q\delta^o_s + \frac{b}{2} B\delta^q_s\,(1 - \delta^o_q) \qquad (34)$$

and

$$b\beta_o(A\delta^o_\ell - A_o)\delta^o_s + (-1)^s \frac{\pi}{a} \frac{\alpha_\ell A'}{\alpha_\ell^2 - \beta_s^2}(1 - \delta^o_\ell) + (-1)^{s+1} \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{\alpha_n A'_n}{\alpha_n^2 - \beta_s^2}$$

$$= b\beta_s B_s\delta^o_s + \frac{b}{2}\beta_s B_s\,(1 - \delta^o_s) - b\beta_s B\delta^o_q\delta^o_s - \frac{b}{2}\beta_s B\delta^q_s\,(1 - \delta^o_q) \qquad (35)$$

$$(s = 0, 1, 2, \ldots)$$

By definition, $A'_n = nA_n \sin \frac{\pi nc}{a}$ and $A' = \ell A \sin \frac{\pi \ell c}{a}$.

The Kronecker Delta δ^b_a is defined as

$$\delta^b_a = 1 \text{ if } a = b$$
$$= 0 \text{ if } a \neq b$$

A similar result can be derived from Equations (32) and (33). Thus,

$$c(A\delta^o_\ell + A_o)\,\delta^o_s + (-1)^{s+1} \frac{\pi}{a} \frac{A'}{\alpha_\ell^2 - \gamma_s^2}(1 - \delta^o_\ell) + (-1)^{s+1} \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n^2 - \gamma_s^2}$$

$$= cC_s\delta^o_s + \frac{c}{2} C_s(1 - \delta^o_s) + cC\delta^o_r\delta^o_s + \frac{c}{2} C\,\delta^r_s\,(1 - \delta^o_r) \qquad (36)$$

* If a/b is a rational number, some of the terms in Equations (34) and (35) will become indeterminate. The equations are correct, however, if the indeterminate forms are replaced by their limits as a/b approaches a rational value.

and

$$c\gamma_o(A\delta^o_\ell - A_o)\delta^o_s + (-1)^{s+1}\frac{\pi}{a}\frac{\alpha_\ell A'_\ell}{\alpha^2_\ell - \gamma^2_s}(1-\delta^o_\ell) + (-1)^s\frac{\pi}{a}\sum_{n=1}^{\infty}\frac{\alpha_n A'_n}{\alpha^2_n - \gamma^2_s}$$

$$= c\gamma_s C_s\delta^o_s + \frac{c}{2}\gamma_s C_s(1-\delta^o_s) - c\gamma_s C\delta^o_r\delta^o_s - \frac{c}{2}\gamma_s C\delta^r_s(1-\delta^o_r) \quad (37)$$

$$(s = 0, 1, 2, \ldots)$$

Now refer to Equations (34) and (35). If each line of set Equation (34) is multiplied by β_s and then sets (34) and (35) are added and subtracted, two alternate sets of infinite equations are derived. Thus, these can be written as

$$b\beta_o A\delta^o_s\delta^o_\ell + (-1)^{s+1}\frac{\pi}{a}\frac{A'}{\alpha_\ell - \beta_s}(1-\delta^o_\ell) + (-1)^s\frac{\pi}{a}\sum_{n=1}^{\infty}\frac{A'_n}{\alpha_n - \beta_s}$$

$$= \frac{b}{2}\beta_s B_s(1-\delta^o_s) + b\beta_o B_o\delta^o_s \quad (38)$$

and

$$-b\beta_o A_o\delta^o_s + (-1)^{s+1}\frac{\pi}{a}\frac{A'}{\alpha_\ell + \beta_s}(1-\delta^o_\ell) + (-1)^s\frac{\pi}{a}\sum_{n=1}^{\infty}\frac{A'_n}{\alpha_n - \beta_s}$$

$$= -\frac{b}{2}\beta_s B\delta^q_s(1-\delta^o_q) - b\beta_o B\delta^o_q\delta^o_s \quad (39)$$

$$(s = 0, 1, 2, \ldots)$$

Similarly, from the sets of Equations (36) and (37), another two sets of equations can be derived. They are

$$c\gamma_o A\delta^o_s\delta^o_\ell + (-1)^s\frac{\pi}{a}\frac{A'}{\alpha_\ell - \gamma_s}(1-\delta^o_\ell) + (-1)^{s-1}\frac{\pi}{a}\sum_{n=1}^{\infty}\frac{A'_n}{\alpha_n + \gamma_s}$$

$$\frac{c}{2}\gamma_s C_s(1-\delta^o_s) \quad c\gamma_o C_o\delta^o_s \quad (40)$$

and

$$-c\gamma_o A_o \delta_s^o + (-1)^s \frac{\pi}{a} \frac{A'}{\alpha_\ell + \gamma_s}(1 - \delta_\ell^o) + (-1)^{s+1} \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n - \gamma_s}$$

$$= - \frac{c}{2} \gamma_s C \delta_s^r (1 - \delta_r^o) - c\gamma_s C \delta_r^o \delta_s^o \tag{41}$$

$$(s = 0, 1, 2, \ldots)$$

3.2 The Solution of the Systems of Equations

In this section, the scattering coefficients are derived. This involves the solution of the systems of equations derived in Section 3.1. Two separate cases are considered. They are a) $A = C = 0$ but $B \neq 0$; and b) $B = C = 0$ but $A \neq 0$. Recall that A, B, and C are the mode amplitudes of the arbitrary modes incident in the three regions as defined in Section 3.1. The solution of the systems of Equations (35) and (37) with $A = C = 0$ gives the elements of the scattering matrices S^{BB}, S^{CB}, and S^{AB} in terms of the mode amplitudes B_m, C_m, and A_m. Explicitly, these are: $S_{mn}^{BB} = - B_m/B$, $S_{mn}^{CB} = - C_m/B$, and $S_{mn}^{AB} = A_m/B$. The electric field or voltage sign convention is followed when defining the scattering coefficients as explained in the previous section. Similarly, the solution of the systems of Equations (35) and (37), with $B = C = 0$ gives the elements of S^{AA}, S^{BA}, and S^{CA} in terms of the mode amplitudes of that problem. These are expressible as: $S_{mn}^{AA} = - A_m/A$, $S_{mn}^{BA} = B_m/A$, and $S^{CA} = C_m/A$.

It is not necessary to go through the formal solution of a system of equations to find the elements of S^{CC}, S^{BC}, and S^{AC} since they follow by a simple transformation from the elements of S^{BB}, S^{CB}, and S^{AB}, respectively. If the dimensions b and c are interchanged in the expressions for S_{mn}^{BB} and S_{mn}^{CB}, then the resulting expressions are identical to S_{mn}^{CC} and S_{mn}^{BC}, respectively. Similarly, if the dimensions b and c are interchanged in the expression for

S^{AB}_{mn} and then the total expression is multiplied by a factor of $(-1)^m$, the resulting expression is identical with S^{AC}_{mn}.

3.2.1 Derivation of the Elements of S^{BB}, S^{CB}, and S^{AB}

With A and C set equal to zero, Equations (35) and (37) can be written as

$$\sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n - \beta_s} + (-1)^s \frac{ab}{2\pi} \beta_s B \delta^q_s \delta^o_q + \frac{ab\beta_o}{\pi} (A_o - B\delta^o_q)\delta^o_s = 0 \tag{42}$$

and

$$\sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n - \gamma_s} - \frac{ac\gamma_o}{\pi} A_o \delta^o_s = 0 \qquad (s = 0, 1, 2, \ldots) \tag{43}$$

where q can be any integer.

The above set of equations can be solved by the function-theoretic technique. A general discussion of the function theoretic-technique is given by Collin[12]. A meromorphic function $f(\omega)$ is constructed in such a manner that it will generate an infinite set of equations which is formally identical with the original set of equations when it is integrated around the correct contour. The form of $f(\omega)$ depends on whether q is zero or non-zero. To simplify the discussion, consider first that $q = 0$, i.e., a TEM mode is used to excite region B.

A $f(\omega)$ is desired such that

$$\lim_{L_n \to \infty} \frac{1}{2\pi j} \oint_{L_n} \frac{f(\omega)\, d\omega}{\omega - \beta_s} = \sum_{n=1}^{\infty} \frac{r(\alpha_n)}{\alpha_n - \beta_s} = f(\beta_o)\delta^o_s \tag{44}$$

where $r(\alpha_n)$ is the residue of $f(\omega)$ at the pole $\omega = \alpha_n$, and

$$f(\beta_o) = \frac{ab\beta_o}{\pi} (A_o - B) \tag{45}$$

and also,

$$\lim_{L_n \to \infty} \frac{1}{2\pi j} \oint_{L_n} \frac{f(\omega)\, d\omega}{\omega - \gamma_s} = \sum_{n=1}^{\infty} \frac{r(\alpha_n)}{\alpha_n - \gamma_s} + f(\gamma_o)\delta_s^o \tag{46}$$

where

$$f(\gamma_o) = -\frac{ac\gamma_o}{\pi} A_o \tag{47}$$

The contour L_n is illustrated in Figure 8. The function $f(\omega)$ is a function with simple poles located at

$$\omega = \alpha_n, \; n = 1, 2, \ldots \tag{48}$$

and simple zeroes at

$$\omega = \gamma_n, \; n = 1, 2, \ldots \tag{49}$$

$$\omega = \beta_n, \; n = 1, 2, \ldots \tag{50}$$

Furthermore, by comparing Equations (44) and (46) with Equations (42) and (43), it is observed that

$$A_n = r(\alpha_n)/n \sin \frac{\pi nc}{a} \tag{51}$$

The function $f(\omega)$ is now determined to within some integral function $p(\omega)$. The function $p(\omega)$ can be determined by examining the asymptotic behaviour of $f(\omega)$ as $\omega \to \infty$. It can be shown that in order for the edge condition[4] to be satisfied,

$$f(\omega) = O(\omega^{-1/2}), \; \omega \to \infty \tag{52}$$

excluding the poles and zeroes on the negative imaginary axis.

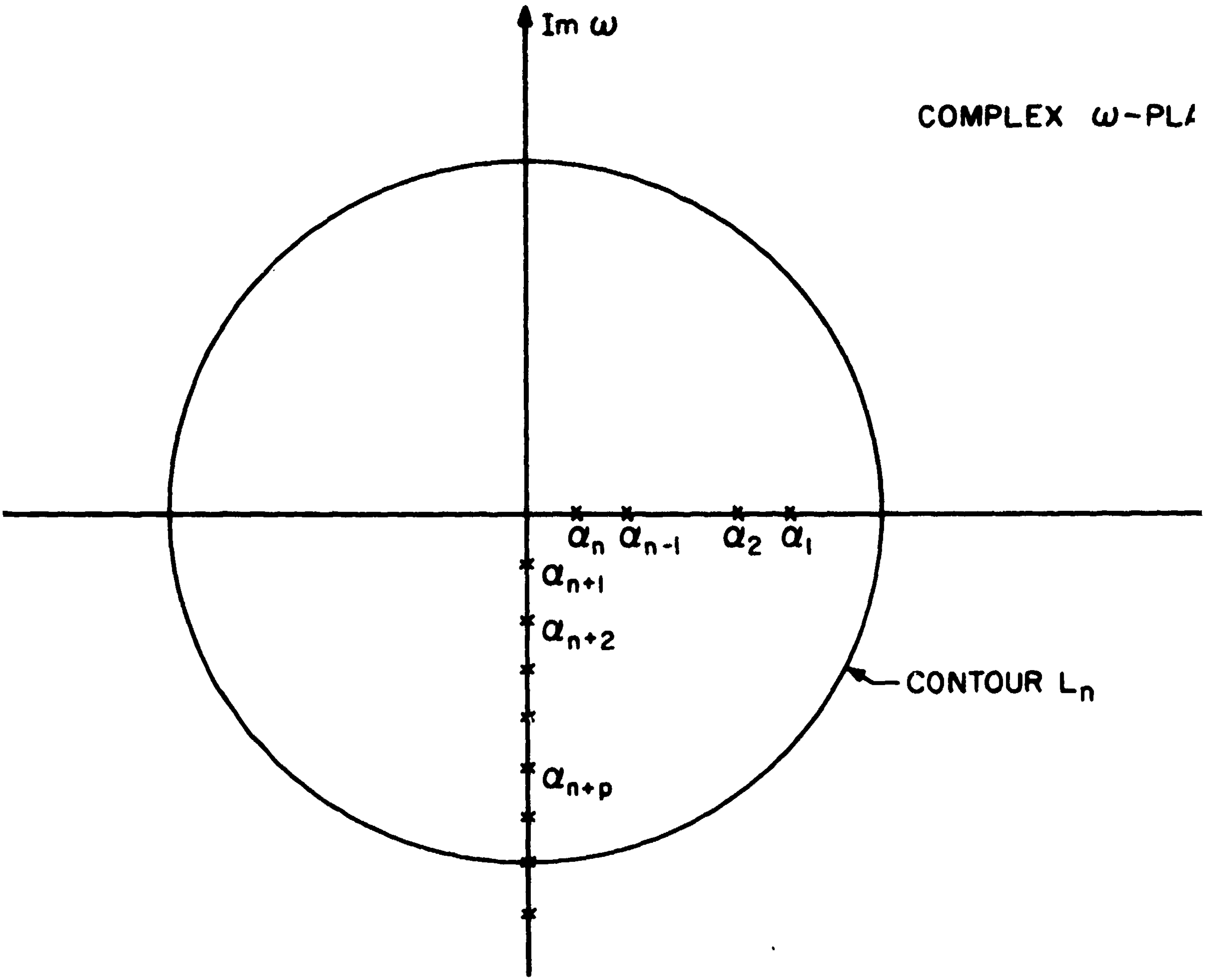


Figure 8. Location of poles of $f(\omega)$ and the contour L_n in the complex ω-plane.

From the above, one result is immediate. Noting that $\beta_o = \gamma_o = k$, Equations (45) and (47) yield $A_o/B = b/a$. By definition, one can write

$$S_{oo}^{AB} = b/a \tag{53}$$

The function $f(\omega)$ can be constructed in the following manner. A function $f(\omega)$ with the necessary poles and zeroes can be written as

$$f(\omega) = p(\omega) \frac{\Pi(\omega,\beta)\ \Pi(\omega,\gamma)}{\Pi(\omega,\alpha)} \tag{54}$$

where, for instance,

$$\Pi(\omega,\beta) = \prod_{n=1}^{\infty} (\beta_n - \omega) \left(\frac{jb}{\pi n}\right) e^{j\frac{b\omega}{n\pi}}$$

The notation is that used by Hurd[7,23] and Whitehead[4]. The inclusion of the exponential terms in the infinite product assures the uniform convergence of the individual products. The function $p(\omega)$, of course, is the integral function mentioned above.

Observe that asymptotically for large n, $\beta_n \sim -j\frac{\pi n}{b}$, $\alpha_n \sim -j\frac{\pi n}{c}$, and $\alpha_n \sim -j\frac{\pi n}{a}$. It can be shown that if $K(\omega)$ is a slowly varying function of ω, then one can write

$$f(\omega) = K(\omega)p(\omega) \frac{\prod_{n=1}^{\infty}(1 - j\frac{\omega c}{\pi n})e^{j\frac{\omega c}{n\pi}} \prod_{n=1}^{\infty}(1 - j\frac{\omega b}{\pi n})e^{j\frac{\omega b}{n\pi}}}{\prod_{n=1}^{\infty} (1 - j\frac{\omega a}{\pi n})e^{j\frac{\omega a}{n\pi}}} \tag{55}$$

The function $f(\omega)$ can be written in terms of the Gamma function. First recall the identity[24]

$$\prod_{n=1}^{\infty} (1 + \frac{u}{n})\ e^{-\frac{u}{n}} = \frac{e^{-\gamma u}}{u\Gamma(u)} \tag{56}$$

where u is a complex variable, excluding the negative integers, γ is "Euler's

Constant", and $\Gamma(u)$ is the Gamma function of argument u. With the aid of this identity, Equation (55) can be written as

$$f(\omega) = j\,\frac{p(\omega)K(\omega)\ \pi a\ \ \Gamma(-j\frac{\omega a}{\pi})}{\omega\ b\ c\ \ \Gamma(-j\frac{\omega b}{\pi})\ \Gamma(-j\frac{\omega c}{\pi})} \tag{57}$$

The function $f(\omega)$ is now written in a form convenient for the examination of its asymptotic behaviour for large values of ω.

Stirling's formula[24] for the asymptotic behaviour of the Gamma function for large argument is given by

$$\Gamma(u) \sim (2\pi)^{1/2}\ e^{(u\ -\ 1/2)\ell n\ u\ -\ u}\ ;\quad u \longrightarrow \infty \tag{58}$$

This is valid everywhere in the complex u-plane except in the vicinity of the negative real axis. Using Stirling's formula, the asymptotic behaviour of $f(\omega)$ can be found to be

$$f(\omega) \sim \left(\frac{ja}{2bc}\right)^{1/2} \frac{p(\omega)K}{\omega^{1/2}}\ e^{j\frac{\omega a}{\pi}\left\{\ell n\left(\frac{a}{b}\right)\ +\ \frac{c}{a}\ \ell n\left(\frac{b}{c}\right)\right\}},\quad \omega \longrightarrow \infty \tag{59}$$

excluding the poles and zeros of K. Thus, in order for $f(\omega)$ to exhibit algebraic growth at infinity

$$p(\omega) \propto e^{-j\frac{\omega a}{\pi}\left\{\ell n\left(\frac{a}{b}\right)\ +\ \frac{c}{a}\ \ell n\left(\frac{b}{c}\right)\right\}} \tag{60}$$

The constant of proportionality is determined by setting $f(\beta_o) = -\frac{cb}{\pi}\beta_o B$. The relationship follows from Equations (50) and (51). Finally, $f(\omega)$ can be written

$$f(\omega) = -\frac{bc\beta_o}{\pi}\ B\ \frac{\Pi(\omega,\beta)\ \Pi(\omega,\gamma)\ \Pi(\beta_o,\alpha)}{\Pi(\beta_o,\beta)\ \Pi(\beta_o,\gamma)\ \Pi(\omega,\alpha)}\ e^{-j\,\frac{(\beta_o-\omega)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right)\ +\ \frac{c}{a}\ \ell n\left(\frac{b}{c}\right)\right\}} \tag{61}$$

C_s and B_S can be found from $f(\omega)$. Note that $f(\omega)$ satisfies

$$\lim_{L_n \to \infty} \frac{1}{2\pi j} \oint_{L_n} \frac{f(\omega)\, d\omega}{\omega + \beta_n} = \sum_{n=1}^{\infty} \frac{r(\alpha_n)}{\alpha_n + \beta_s} + f(-\beta_s) \tag{62}$$

and

$$\lim_{L_n \to \infty} \frac{1}{2\pi j} \oint_{L_n} \frac{f(\omega)\, d\omega}{\omega + \gamma_s} = \sum_{n=1}^{\infty} \frac{r(\alpha_n)}{\alpha_n + \gamma_s} + f(-\gamma_s) \tag{63}$$

L_n is the same contour as befor . Equations (62) and (63) are formally identical with Equations (37) and (39). It follows that

$$B_s = (-1)^s \frac{2\pi}{\epsilon_s ab} f(-\beta_s) \tag{64}$$

and

$$C_s = (-1)^{s+1} \frac{2\pi}{\epsilon_s ac} f(-\gamma_s) \tag{65}$$

where $\epsilon_s = \begin{cases} 2 \text{ if } s = 0 \\ 1 \text{ if } s \neq 0 \end{cases}$

It can be shown that for other modes of excitation, Equations (64) and (65) are valid providing, of course, the proper $f(\omega)$ is used in the calculations.

S^{BB}_{po}, S^{CB}_{po}, and S^{AB}_{po} follow immediately from Equations (61), (64), and (65) and the definitions of the scattering coefficients. Thus, for all p,

$$S^{BB}_{po} = \frac{(-1)^p 2\beta_o b\, \Pi(-\beta_p, \beta)\, \Pi(-\beta_p, \gamma)\, \Pi(\beta_o, \alpha)}{\epsilon_p \beta_p a\, \Pi(\beta_o, \beta)\, \Pi(\beta_o, \gamma)\, \Pi(-\beta_p, \alpha)} e^{-j \frac{(\beta_o + \beta_p)}{\pi} a_L} \tag{66}$$

and

$$S^{BC}_{po} = \frac{(-1)^{p+1} 2\beta_o b\, \Pi(-\gamma_p, \beta)\, \Pi(-\gamma_p, \gamma)\, \Pi(\beta_o, \alpha)}{\epsilon_p \gamma_p a\, \Pi(\beta_o, \beta)\, \Pi(\beta_o, \gamma)\, \Pi(-\gamma_p, \alpha)} e^{-j \frac{(\beta_o + \gamma_p)}{\pi} a_L} \tag{67}$$

and for p greater than zero,

$$S^{BA}_{po} = \frac{-bc\beta_o}{\pi p \sin\left(\frac{\pi p c}{a}\right)} \frac{\Pi(\alpha_p, \beta)\, \Pi(\alpha_p, \gamma)\, \Pi(\beta_o, \alpha)}{\Pi(\beta_o, \beta)\, \Pi(\beta_o, \gamma)\, \Pi^{(p)}(\alpha_p, \alpha)} \, e^{-j \frac{(\beta_o - \alpha_p)}{\pi} a_L} \tag{68}$$

Here the superscript p in the infinite product means delete the term $(\alpha_n - \alpha_p)$, $n = p$, from the product. By definition, $a_L = a\left\{ \ln\left(\frac{a}{c}\right) + \frac{b}{a} \ln\left(\frac{c}{b}\right) \right\}$.

Consider now the case of a higher order TM mode in region B used to excite the bifurcated waveguide. The procedure for the solution of Equations (42) and (43) is essentially the same as before, differing only in its details. First another meromorphic function of the complex variable ω is constructed. Call this function $f_1(\omega)$. This is a function with simple poles located at

$$\omega = \alpha_n, \quad n = 1, 2, \ldots \tag{69}$$

and simple zeros at

$$\omega = \gamma_n, \quad n = 1, 2, \ldots \tag{70}$$

and

$$\omega = \beta_n, \quad n = 0, 1, 2, \ldots \tag{71}$$

except at $\omega = \beta_q$. Furthermore,

$$f(\beta_q) = (-1)^q \frac{ab}{2\pi} \beta_q B \tag{72}$$

Again, the asymptotic behaviour of the function at infinity must be studied, and in order to satisfy the edge condition, $f_1(\omega) = O(\omega^{-1/2})$ as $\omega \longrightarrow \infty$.

Note that in this case, $f_1(\omega)$ has a zero at $\omega = \beta_o$. This is because a TEM mode is not excited in region A by a higher order TM excited in region B, i.e., $A_o = 0$. This follows immediately if the concept of reciprocity is applied. If a TEM mode is excited in region A, no higher order modes will be

scattered or reflected by the bifurcation. This is because the incident TEM mode is already a normal mode of the system. Hence, if no higher order mode is excited in region B by an incident TEM mode in region A, by reciprocity, no TEM mode will be excited in region A by an incident higher order mode in region B. By definition, for $q > 0$,

$$S^{AB}_{oq} = 0 \tag{73}$$

The proper $f_1(\omega)$ can be shown to be

$$f_1(\omega) = \frac{(-1)^q ba\, E\, \beta_q (\omega - \beta_o)\, \Pi(\omega,\beta)\, \Pi(\omega,\gamma)\, \Pi(\beta_q,\alpha)}{2\pi(\beta_q - \omega)(\beta_q - \beta_o)\, \Pi^{(q)}(\beta_q,\beta)\, \Pi(\beta_q,\gamma)\, \Pi(\omega,\alpha)}\, e^{-j\frac{(\beta_q-\omega)}{\pi} a_L} \tag{74}$$

From Equations (64), (65), and (74) and the definitions of the scattering coefficients, one finds for $q > 0$,

$$S^{BB}_{pq} = \frac{(-1)^{p+q+1}\beta_q(\beta_p+\beta_o)\, \Pi(-\beta_p,\beta)\, \Pi(-\beta_p,\gamma)\, \Pi(\beta_q,\alpha)}{\epsilon_p\beta_p(\beta_p+\beta_q)(\beta_q-\beta_o)\, \Pi^{(q)}(\beta_q,\beta)\, \Pi(\beta_q,\gamma)\, \Pi(-\beta_p,\alpha)}\, e^{-j\frac{(\beta_p+\beta_q)}{\pi} a_L} \tag{75}$$

and

$$S^{CB}_{pq} = \frac{(-1)^{p+q} b\beta_q(\gamma_p+\beta_o)\, \Pi(-\gamma_p,\beta)\, \Pi(-\gamma_p,\gamma)\, \Pi(\beta_q,\alpha)}{\epsilon_p c\gamma_p(\gamma_p+\beta_q)(\beta_q-\beta_o)\, \Pi^{(q)}(\beta_q,\beta)\, \Pi(\beta_q,\gamma)\, \Pi(-\gamma_q,\alpha)}\, e^{-j\frac{(\gamma_p+\beta_q)}{\pi} a_L} \tag{76}$$

and for $p > 0$, $q > 0$,

$$S^{AB}_{pq} = \frac{(-1)^q ca\beta_q(\alpha_p-\beta_o)\, \Pi(\alpha_p,\beta)\, \Pi(\alpha_p,\gamma)\, \Pi(\beta_q,\alpha)}{2\pi p \sin\left(\frac{\pi pc}{a}\right)(\beta_q-\alpha_p)(\beta_q-\beta_o)\Pi^{(q)}(\beta_q,\beta)\Pi(\beta_q,\gamma)\Pi^{(p)}(\alpha_p,\alpha)}\, e^{-j\frac{(\alpha_p-\beta_q)}{\pi} a_L} \tag{77}$$

Note that at the beginning of this section, it was assumed that the mode incident in any of the three regions was a propagating mode. The derivation

of the expressions for the scattering coefficients was based on this assumption. It is necessary, also, that the scattering coefficients be defined if the incident mode is evanescent. To do this, it is only necessary in the expressions for the scattering coefficients to replace the appropriate real propagation constants representing the propagating modes by the corresponding imaginary propagation constants representing evanescent modes. The expressions derived above are tabulated in Table 1.

3.2.2 Derivation of the Elements of S^{AA}, S^{BA}, and S^{CA}

With B and C set equal to zero, Equations (39) and (41) can be written as

$$b\beta_o A_o \delta^o_s + (-1)^{s+1} \frac{\pi}{a} \frac{A'}{\alpha_\ell + \beta_s} (1 - \delta^o_\ell) + (-1)^s \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n - \beta_s} = 0 \quad (78)$$

and

$$c\gamma_o A_o \delta^o_s + (-1)^s \frac{\pi}{a} \frac{A'}{\alpha_\ell + \gamma_s} (1 - \delta^o_\ell) + (-1)^{s+1} \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{A'_n}{\alpha_n - \gamma_s} = 0 \quad (79)$$

$$(s = 0, 1, 2, \ldots)$$

As already noted in Section 3.2.1, a TEM mode excited in region A will not be reflected by the bifurcation nor will any higher order modes be excited. Hence, for all p, one can write

$$S^{AA}_{po} = 0 \quad (80)$$

and for $q > 0$,

$$S^{AA}_{oq} = 0 \quad (81)$$

Furthermore,

$$S^{BA}_{oo} = 1 \quad (82)$$

TABLE 1

ELEMENTS OF S^{BB}, S^{CB}, AND S^{AB}

$$S^{BB} = \begin{bmatrix} S^{BB}_{po} & S^{BB}_{pq} \end{bmatrix}$$

$$S^{BB}_{po} = \frac{(-1)^{p}\, 2\beta_o c}{\epsilon_p\, \beta_p\, a}\, \frac{\Pi(-\beta_p,\beta)\, \Pi(-\beta_p,\gamma)\, \Pi(\beta_o,\alpha)}{\Pi(\beta_o,\beta)\, \Pi(\beta_o,\gamma)\, \Pi(-\beta_p,\alpha)}\, e^{-j\frac{(\beta_o + \beta_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

$$S^{BB}_{pq} = \frac{(-1)^{p+q+1}\, \beta_q(\beta_p + \beta_o)\, \Pi(-\beta_p,\beta)\, \Pi(-\beta_p,\gamma)\, \Pi(\beta_q,\alpha)}{\epsilon_p\beta_p(\beta_p + \beta_q)(\beta_q - \beta_o)\, \Pi^{(q)}(\beta_q,\beta)\, \Pi(\beta_q,\gamma)\, \Pi(-\beta_p,\alpha)}\, e^{-j\frac{(\beta_p + \beta_q)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

$$S^{CB} = \begin{bmatrix} S^{CB}_{po} & S^{CB}_{pq} \end{bmatrix}$$

$$S^{CB}_{po} = \frac{(-1)^{p+1}\, 2\beta_o b}{\epsilon_p\gamma_p\, a}\, \frac{\Pi(-\gamma_p,\beta)\, \Pi(-\gamma_p,\gamma)\, \Pi(\beta_q,\alpha)}{\Pi(\beta_o,\beta)\, \Pi(\beta_o,\gamma)\, \Pi(-\gamma_p,\alpha)}\, e^{-j\frac{(\beta_o + \gamma_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

$$S^{CB}_{pq} = \frac{(-1)^{p+q}\quad b\, \beta_q(\gamma_p + \beta_o)\, \Pi(-\gamma_p,\beta)\, \Pi(-\gamma_p,\gamma)\, \Pi(\beta_q,\alpha)}{\epsilon_p\, c\, \gamma_p(\gamma_p + \beta_q)(\beta_q - \beta_o)\, \Pi^{(q)}(\beta_q,\beta)\, \Pi(\beta_q,\gamma)\, \Pi(-\gamma_p,\alpha)}\, e^{-j\frac{(\beta_q + \gamma_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

$$S^{AB}_{oo} = b/a \qquad S^{AB}_{oq} = 0 \qquad\qquad S^{AB} = \begin{bmatrix} S^{AB}_{oo} & S^{AB}_{oq} \\ S^{AB}_{po} & S^{AB}_{pq} \end{bmatrix}$$

$$S^{AB}_{po} = \frac{-bc\beta_o}{\pi p \sin\frac{\pi pc}{a}}\, \frac{\Pi(\alpha_p,\beta)\, \Pi(\alpha_p,\gamma)\, \Pi(\beta_o,\alpha)}{\Pi(\beta_o,\beta)\, \Pi(\beta_o,\gamma)\, \Pi^{(p)}(\alpha_p,\alpha)}\, e^{-j\frac{(\beta_o - \alpha_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

$$S^{AB}_{pq} = \frac{(-1)^{q}\, c\, a\, \beta_q(\alpha_p - \beta_o)\, \Pi(\alpha_p,\beta)\, \Pi(\alpha_p,\beta)\, \Pi(\beta_q,\alpha)}{2\pi\, p \sin\frac{\pi pc}{a}\, (\beta_q - \alpha_p)(\beta_q - \beta_o)\, \Pi^{(q)}(\beta_q,\beta)\, \Pi(\beta_q,\gamma)\, \Pi^{(p)}(\alpha_p,\alpha)}\, e^{-j\frac{(\beta_q - \alpha_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

S^{AB}

$\frac{)a}{} \left\{ \ln\left(\frac{a}{b}\right) + \frac{c}{a} \ln\left(\frac{b}{c}\right)\right\}$ $\frac{}{a)} e^{-j\frac{(\beta_p + \beta_q)a}{\pi}\left\{ \ln\left(\frac{a}{b}\right) + \frac{c}{a} \ln\left(\frac{b}{c}\right)\right\}}$
$\frac{\gamma_p)a}{}\left\{ \ln\left(\frac{a}{b}\right) + \frac{c}{a} \ln\left(\frac{b}{c}\right)\right\}$ $\frac{)}{\gamma_p, a)} e^{-j\frac{(\beta_q + \gamma_p)a}{\pi}\left\{ \ln\left(\frac{a}{b}\right) + \frac{c}{a} \ln\left(\frac{b}{c}\right)\right\}}$
$\frac{p)a}{}\left\{ \ln\left(\frac{a}{b}\right) + \frac{c}{a} \ln\left(\frac{b}{c}\right)\right\}$ $\frac{}{\gamma)\ \Pi^{(p)}(\alpha_p, \alpha)} e^{-j\frac{(\beta_q - \alpha_p)a}{\pi}\left\{ \ln\left(\frac{a}{b}\right) + \frac{c}{a} \ln\left(\frac{b}{c}\right)\right\}}$

and

$$S_{oo}^{CA} = 1 \tag{83}$$

Only if a higher order TM mode in region A excites the waveguide, does the system of Equations (78) and (79) have non-trivial solutions.

Again, the function-theoretic technique is called upon to solve the above equations. A function, say $f_2(\omega)$, is constructed such that when it is integrated along contour L_n, it generates a system of equations which are formally identical with Equations (78) and (79), $\ell \neq 0$. $f_2(\omega)$ is found to be

$$f_2(\omega) = \frac{-A'(\omega-\beta_o)\,\Pi(\omega,\beta)\,\Pi(\omega,\gamma)\,\Pi(-\alpha_\ell,\alpha)}{(\alpha_\ell+\beta_o)(\omega+\alpha_\ell)\,\Pi(-\alpha_\ell,\beta)\,\Pi(-\alpha_\ell,\gamma)\,\Pi(\omega,\alpha)}\, e^{j\frac{(\omega+\alpha_\ell)}{\pi}a_L} \tag{84}$$

A_p' is the residue of $f_2(\omega)$ at $\omega = \alpha_p$ so that

$$S_{pq}^{AA} = \left(\frac{q\sin\frac{\pi qc}{a}}{p\sin\frac{\pi pc}{a}}\right)\frac{(\alpha_p-\beta_o)\,\Pi(\alpha_p,\beta)\,\Pi(\alpha_p,\gamma)\,\Pi(-\alpha_q,\alpha)}{(\alpha_q+\beta_o)(\alpha_p+\alpha_q)\Pi(-\alpha_q,\beta)\Pi(-\alpha_q,\gamma)\Pi^{(p)}(\alpha_p,\alpha)}\, e^{j\frac{(\alpha_p+\alpha_q)}{\pi}a_L} \tag{85}$$

for $p, q > 0$.

Furthermore, using Equations (64) and (65), one finds for $p, q > 0$,

$$S_{pq}^{BA} = \frac{(-1)^{p+1}2\pi q\sin\frac{\pi qc}{a}\,(\beta_p+\beta_o)\Pi(-\beta_p,\beta)\Pi(-\beta_p,\gamma)\Pi(-\alpha_q,\alpha)}{ab(\alpha_q+\beta_o)(\alpha_q-\beta_p)\Pi(-\alpha_q,\beta)\Pi(-\alpha_q,\gamma)\Pi(-\beta_p,\alpha)}\, e^{j\frac{(\alpha_q-\beta_p)}{\pi}a_L} \tag{86}$$

and

$$S_{pq}^{AC} = \frac{(-1)^{p}2\pi q\sin\left(\frac{\pi qc}{a}\right)(\gamma_p+\beta_o)\Pi(-\gamma_p,\beta)\Pi(-\gamma_p,\gamma)\Pi(-\alpha_q,\alpha)}{ac(\alpha_q+\beta_o)(\alpha_q-\gamma_p)\Pi(-\alpha_q,\beta)\Pi(-\alpha_q,\gamma)\Pi(-\gamma_p,\alpha)}\, e^{j\frac{(\alpha_q-\gamma_p)}{\pi}a_L} \tag{87}$$

The elements of S^{AA}, S^{BA} and S^{CA} are tabulated in Table 2. As before, to get the scattering coefficients for evanescent incident modes, simply re-

TABLE 2

ELEMENTS OF S^{AA}, S^{BA}, AND S^{CA}

$$S^{AA}_{oo} = 0 \qquad S^{AA}_{po} = 0 \qquad S^{AA}_{oq} = 0 \qquad S^{AA} = \begin{bmatrix} S^{AA}_{oo} & S^{AA}_{oq} \\ S^{AA}_{po} & S^{AA}_{pq} \end{bmatrix}$$

$$S^{AA}_{pq} = \frac{q \sin \frac{\pi q c}{a}}{p \sin \frac{\pi p c}{a}} \; \frac{(\alpha_p - \beta_o)\, \Pi(\alpha_p, \beta)\, \Pi(\alpha_p, \gamma)\, \Pi(-\alpha_q, \alpha)}{(\alpha_q + \beta_o)(\alpha_p + \alpha_q)\, \Pi(-\alpha_q, \beta)\, \Pi(-\alpha_q, \gamma)\, \Pi^{(p)}(\alpha_p, \alpha)} \; e^{j\frac{(\alpha_p + \alpha_q)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\right.}$$

$$S^{BA}_{oo} = 1 \qquad S^{BA}_{po} = 0 \qquad S^{BA} = \begin{bmatrix} S^{BA}_{oo} & S^{BA}_{oq} \\ S^{BA}_{po} & S^{BA}_{pq} \end{bmatrix}$$

$$S^{BA}_{pq} = \frac{2\pi\, q \sin \frac{\pi q c}{a}\, (\beta_p + \beta_o)\, \Pi(-\beta_p, \beta)\, \Pi(-\beta_p, \gamma)\, \Pi(-\alpha_q, \alpha)}{\epsilon_p\, ab\, (\alpha_q + \beta_o)(\alpha_q - \beta_p)\, \Pi(-\alpha_q, \beta)\, \Pi(-\alpha_q, \gamma)\, \Pi(-\beta_q, \alpha)} \; e^{j\frac{(\alpha_q - \beta_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\frac{b}{c}\right)\right\}}$$

$$S^{CA}_{oo} = 1 \qquad S^{CA}_{po} = 0 \qquad S^{CA} = \begin{bmatrix} S^{CA}_{oo} & S^{CA}_{oq} \\ S^{CA}_{po} & S^{CA}_{pq} \end{bmatrix}$$

$$S^{CA}_{pq} = \frac{(-1)^{p+1}\, 2\pi\, q \sin \frac{\pi q c}{a}\, (\gamma_p + \beta_o)\, \Pi(-\gamma_p, \beta)\, \Pi(-\gamma_p, \gamma)\, \Pi(-\alpha_q, \alpha)}{\epsilon_p\, ac\, (\alpha_q + \beta_o)(\alpha_q - \gamma_p)\, \Pi(-\alpha_q, \beta)\, \Pi(-\alpha_q, \gamma)\, \Pi(-\gamma_p, \alpha)} \; e^{j\frac{(\alpha_q - \gamma_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right) + \frac{c}{a}\,\ell n\left(\right.\right.}$$

TABLE 2

ELEMENTS OF S^{AA}, S^{BA}, AND S^{CA}

$S^{AA} = \begin{bmatrix} S^{AA}_{oo} & S^{AA}_{oq} \\ S^{AA}_{po} & S^{AA}_{pq} \end{bmatrix}$ $\dfrac{,\beta)\ \Pi(\alpha_p,\gamma)\ \Pi(-\alpha_q,\alpha)}{\alpha_q)\ \Pi(-\alpha_q,\beta)\ \Pi(-\alpha_q,\gamma)\ \Pi^{(p)}(\alpha_p,\alpha)}\ e^{j\frac{(\alpha_p+\alpha_q)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right)+\frac{c}{a}\ \ell n\left(\frac{b}{c}\right)\right\}}$
$S^{BA} = \begin{bmatrix} S^{BA}_{oo} & S^{BA}_{oq} \\ S^{BA}_{po} & S^{BA}_{pq} \end{bmatrix}$ $\dfrac{\beta_p,\beta)\ \Pi(-\beta_p,\gamma)\ \Pi(-\alpha_q,\alpha)}{(-\alpha_q,\beta)\ \Pi(-\alpha_q,\gamma)\ \Pi(-\beta_q,\alpha)}\ e^{j\frac{(\alpha_q-\beta_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right)+\frac{c}{a}\ \ell n\left(\frac{b}{c}\right)\right\}}$
$S^{CA} = \begin{bmatrix} S^{CA}_{oo} & S^{CA}_{oq} \\ S^{CA}_{po} & S^{CA}_{pq} \end{bmatrix}$ $\dfrac{\beta_o)\ \Pi(-\gamma_p,\beta)\ \Pi(-\gamma_p,\gamma)\ \Pi(-\alpha_q,\alpha)}{(-\alpha_q,\beta)\ \Pi(-\alpha_q,\gamma)\ \Pi(-\gamma_p,\alpha)}\ e^{j\frac{(\alpha_q-\gamma_p)a}{\pi}\left\{\ell n\left(\frac{a}{b}\right)+\frac{c}{a}\ \ell n\left(\frac{b}{c}\right)\right\}}$

place the appropriate imaginary propagation constants by the corresponding real propagation constants in the given expressions.

3.3 A Note on the Numerical Computations

The exact expressions for the various scattering coefficients are listed in Tables 1 and 2. However, as the expressions stand, they are not in a form convenient for purposes of calculation, even using a digital computer. An infinite product such as $\prod_{n=1}^{\infty} (\beta_n + \beta_p)\left(\frac{jb}{\pi n}\right) e^{-j\frac{b\beta_p}{n\pi}}$ where $\beta_p = -j\sqrt{\left(\frac{\pi p}{b}\right)^2 - k^2}$ is very slowly convergent, especially for large p. However, it is possible to express the various scattering coefficients in a form suitable for calculations, using a digital computer.

As an example, consider S_{pq}^{BB} (see Equation (75)). Assume that the dimensions of the guide are such as to allow only the dominant mode to propagate in each of the three regions A, B, and C. It was indicated in Section 3.1 that it is possible to write S_{pq}^{BB} in terms of the Gamma Function. To explain more fully, consider the product above, which is a term in the expression for S_{pq}^{BB}. It is possible to write

$$\prod_{n=1}^{\infty} (\beta_n + \beta_p) \frac{jb}{\pi n} \epsilon^{j\frac{b\beta_p}{\pi n}} = \prod_{n=1}^{\infty} \left(\frac{j\frac{\beta_n b}{\pi n} + j\frac{\beta_p b}{\pi n}}{1 + j\frac{\beta_p b}{\pi n}} \right) \prod_{n=1}^{\infty} \left(1 + j\frac{\beta_p b}{\pi n}\right) \epsilon^{-j\frac{\beta_p b}{\pi n}} \tag{88}$$

Using Equation (56) in Equation (88), one gets

$$\prod_{n=1}^{\infty} (\beta_n + \beta_p) \left(\frac{jb}{\pi n}\right) \epsilon^{j\frac{b\beta_p}{n\pi}} \prod_{n=1}^{\infty} \left(\frac{j\frac{\beta_n b}{\pi n} + j\frac{\beta_p b}{\pi n}}{1 + j\frac{\beta_p b}{\pi n}} \right) \frac{e^{-j\frac{\gamma\beta_p b}{\pi}}}{\left(j\frac{\beta_p b}{\pi}\right) \Gamma\left(j\frac{\beta_p b}{\pi}\right)} \tag{89}$$

It can be shown [25] that the infinite product on the right side of Equation (89) is convergent. Furthermore, it converges much more rapidly than the

original product. The Gamma Function can be calculated on the digital computer probably using an already available library subroutine.

It follows that S_{pq}^{BB} can be written as

$$S_{pq}^{BB} = \frac{(-1)^{p+q+1} q\left|\beta_q\right|^2(\beta_o - j\left|\beta_p\right|) P\,\Gamma\left(\frac{-\left|\beta_q\right|c}{\pi}\right)\Gamma\left(\frac{-\left|\beta_q\right|b}{\pi}\right)\Gamma\left(\frac{\left|\beta_p\right|a}{\pi}\right)}{\pi\left|\beta_p\right|^2(\left|\beta_p\right| + \left|\beta_q\right|)(\beta_o + j\left|\beta_q\right|)\Gamma\left(\frac{\left|\beta_p\right|c}{\pi}\right)\Gamma\left(\frac{\left|\beta_p\right|b}{\pi}\right)\Gamma\left(\frac{-\left|\beta_q\right|a}{\pi}\right)} e^{\frac{-(\left|\beta_p\right| + \left|\beta_q\right|)}{\pi} a_L} \tag{90}$$

where P is given by

$$P = \prod_{n=1}^{\infty}\left(\frac{\frac{\left|\beta_n\right|b}{\pi n} + \frac{\left|\beta_p\right|b}{\pi n}}{1 + \frac{\left|\beta_p\right|b}{\pi n}}\right)\left(\frac{\frac{\left|\gamma_n\right|c}{\pi n} + \frac{\left|\beta_p\right|c}{\pi n}}{1 + \frac{\left|\beta_p\right|c}{\pi n}}\right)\left(\frac{1 + \frac{\left|\beta_p\right|a}{\pi n}}{\frac{\left|\alpha_p\right|a}{\pi n} + \frac{\left|\beta_p\right|a}{\pi n}}\right) \text{ (times)} \tag{91}$$

$$\prod_{n=1}^{\infty}{}' \left(\frac{1 - \frac{\left|\beta_q\right|b}{n\pi}}{\frac{\left|\beta_n\right|b}{\pi n} - \frac{\left|\beta_q\right|b}{\pi n}}\right)\left(\frac{1 - \frac{\left|\beta_q\right|c}{\pi n}}{\frac{\left|\gamma_n\right|c}{\pi n} - \frac{\left|\beta_q\right|c}{\pi n}}\right)\left(\frac{\frac{\left|\alpha_n\right|a}{\pi n} - \frac{\left|\beta_q\right|a}{\pi n}}{1 - \frac{\left|\beta_q\right|a}{\pi n}}\right)$$

Here the prime $'$ means omit the factor $\left(\frac{\left|\beta_n\right|b}{\pi n} - \frac{\left|\beta_q\right|b}{\pi n}\right)$, $n = q$, from the infinite product.

4. THE INHOMOGENEOUS E-PLANE BIFURCATION IN A PARALLEL PLATE WAVEGUIDE

In this chapter, the generalized scattering matrix technique is used to derive the solution to the first of the three boundary value problems discussed in this thesis, viz., the inhomogeneous E-plane bifurcation in a parallel plate waveguide. Refer to Figure 2. The auxiliary problem is the boundary value problem associated with the bifurcation in a parallel plate waveguide. The bifurcated waveguide is modified by introducing a dielectric slab with a relative dielectric constant in region B. The slab completely fills region B.

With reference to Figure 2, let a TEM mode be incident from region A. The scattered and reflected fields can be represented in terms of TM_{no} modes. As in the previous section, the non-zero field components can be derived from a scalar function identical to H_y. Call this function $\psi(x,z)$. The non-zero field components can be written as

$$H_y = \psi \tag{92}$$

$$E_x = \frac{1}{j\omega\epsilon}\frac{\partial\psi}{\partial z} \tag{93}$$

and

$$E_z = \frac{-1}{j\omega\epsilon}\frac{\partial\psi}{\partial x} \tag{94}$$

where $\epsilon = \epsilon_o$ in regions A and C filled with air and $\epsilon = \kappa\epsilon_o$ in region B filled with dielectric. In regions A and C, ψ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi = 0, \quad k = \omega\sqrt{\mu_o\epsilon_o} \tag{95}$$

In region B, ψ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k'^2\right) \psi = 0, \quad k' = k\sqrt{K} \tag{96}$$

ψ also satisfies the boundary conditions

$$\frac{\partial \psi}{\partial x} = 0, \; x = 0, a, \text{ all } z \text{ and } x = c, \; z > 0 \tag{97}$$

and at the dielectric-air interface,

$$\psi(0^-) = \psi(0^+), \; z = 0, \; c \le x \le a \tag{98}$$

$$\frac{\partial \psi(o^-)}{\partial z} = \frac{\partial \psi(o^+)}{K \, \partial z}, \quad z = 0, \; c \le x \le a \tag{99}$$

ψ also satisfies the edge condition at the edge of the bifurcation, i.e.,

$$|\nabla \psi| = O(d^{-1/2}), \; d \longrightarrow 0 \tag{100}$$

where $d = [z^2 + (x - c)^2]^{1/2}$

One recognizes that the edge condition stated above is the same as the edge condition for a bifurcation without the dielectric present in region B. Figure 9 illustrates the general case of a metallic wedge with a dielectric wedge situated next to it. Using the condition that the electromagnetic energy density must be integrable over any finite domain, Meixner[26] shows that an admissible singularity in $|\nabla \psi|$ at the edge of the composite wedge is given by $|\nabla \psi| = O(d^{-1/2})$ as $d \longrightarrow 0$, where $\eta = \frac{\pi - \xi}{2\pi - \xi}$. If $\xi = 0$, $\eta = 1/2$.

It is possible to show that each individual term in the multiple scattering process as expressed by Equation (1) of Section 2.1 satisfies the above stated edge condition, i.e., the n^{th} partial wave transmitted to

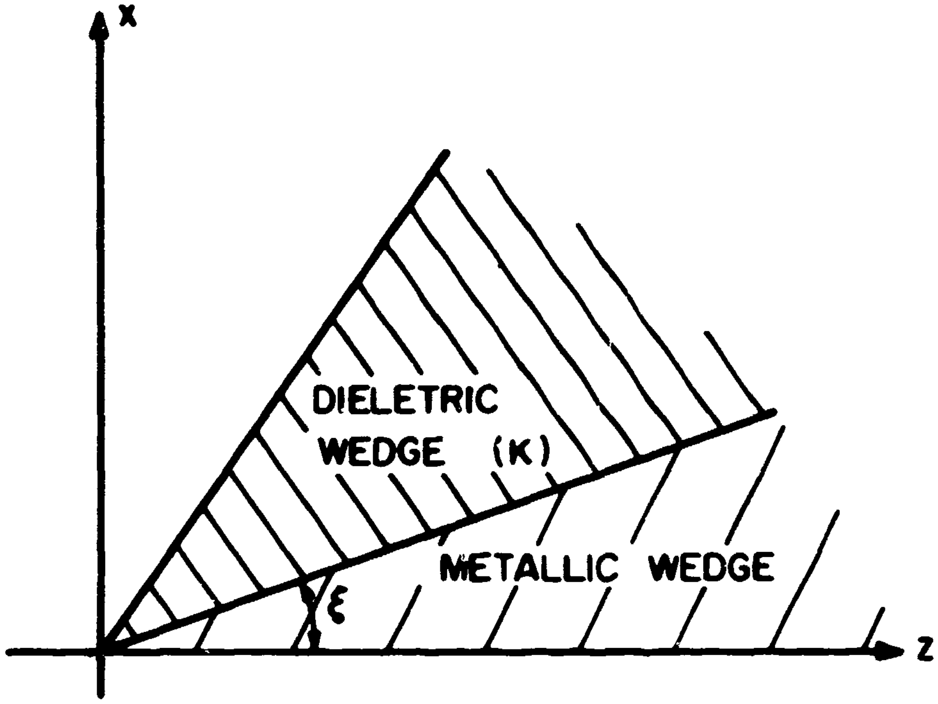


Figure 9. Wedge composed of dielectric and Metallic sections.

region A satisfies the edge condition given by Equation (100). The mode coefficients of the eigenfunction expansion of the n^{th} partial wave in region A are given by the elements of the vector $S^{AB}\Gamma^{B}(S^{BB}\Gamma^{B})^{n}S^{BA}\bar{a}$. The vector $\bar{S}_{in}$ associated with the total field reflected in region A is the sum of the infinite number of multiple reflections. That the order of the singularity at the edge of the bifurcation remains unaltered by the addition of the infinite number of terms in the series given by Equation (1) has not been formally proved. However, the convergence of the series given by (1) was proved in Section 2.2.

It should be noted that the usefulness of a solution to a problem is not necessarily dependent on whether or not the edge condition or any of the boundary conditions are exactly satisfied. Schelkunoff[27] has noted that nearly correct calculations are sometimes possible from solutions which only to a crude approximation satisfy the boundary conditions of the problem.

Without any loss of generality, it can be assumed that the amplitude of the incident TEM mode is unity. Then the total field ψ in region A, denoted by ψ_A, can be written as

$$\psi_A = e^{-j\alpha_o z} - R_A e^{j\alpha_o z} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{a}\right) e^{j\alpha_n z} \tag{101}$$

and, denoting the total field in region C by ψ_C,

$$\psi_C = T_{CA} e^{j\alpha_o z} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right) e^{-j\gamma_n z} \tag{102}$$

The propagation constants α_n and γ_n are defined in Section 3.1. R_A is the voltage reflection coefficient for the dominant mode and T_{CA} is the transmission coefficient from region A to region C. The total field in region B,

denoted by ψ_B, can be written as

$$\psi_B = T_{BA} e^{j\beta'_o z} + \sum_{n=1}^{\infty} B_n \cos\left(\frac{\pi n(x-a)}{b}\right) e^{-j\beta'_n z} \tag{103}$$

where

$$\beta'_n = \sqrt{\kappa k^2 - \left(\frac{\pi n}{b}\right)^2}, \quad \kappa k > \frac{\pi n}{b}$$

$$= -j\sqrt{\left(\frac{\pi n}{b}\right)^2 - \kappa k^2}, \quad \frac{\pi n}{b} > \kappa k$$

T_{BA} is the transmission coefficient for the TEM from region A to region B.

The reflectance and transmission matrices which characterize the dielectric load can be derived by appealing to simple waveguide theory. They are both diagonal matrices. The diagonal elements of the reflectance matrix are given by

$$\rho_n^B = \frac{g'_n - g_n}{g'_n + g_n} \tag{104}$$

where g'_n is the wave impedance of the TM_{no} mode in a parallel plate waveguide, height b, filled with dielectric and g_n is the wave impedance of the TM_{no} mode in the same waveguide filled with air. Specifically,

$$g'_n = \frac{\beta'_n}{\omega\epsilon_o \kappa} \tag{105}$$

and

$$g_n = \frac{\beta_n}{\omega\epsilon_o} \tag{106}$$

Call this reflection matrix Γ_D^B. The transmission matrix is simply related to the reflectance matrix Γ_D^B. Let the transmission matrix be denoted by Φ_D^B.

Then

$$\Phi_D^B = I + \Gamma_D^B \tag{107}$$

The general diagonal element of Φ_D^B is given by

$$\phi_n^B = \frac{2g_n'}{g_n' + g_n} \tag{108}$$

Using Equation (2), it is now possible to write the mode coefficients in Equations (101) in terms of known quantities. Let S'_{in} be defined as the vector $(R_A, -A_1, -A_2, \ldots)^T$. Then

$$S'_{in} = S^{AB}\,\Gamma_D^B\,(I - S^{BB}\,\Gamma_D^B)^{-1}\,S^{BA}\,\bar{a} \tag{109}$$

The elements of the scattering matrices S^{BA}, S^{BB}, and S^{AB} are tabulated in Tables 1 and 2. Equations (101) and (109) completely determine the fields in region A.

Let S'_{CA} be defined as the vector $(T_{CA}, C_1, C_2, \ldots)^T$. Then,

$$S'_{CA} = S^{CA}\bar{a} + S^{CB}\,\Gamma_D^B\,(I - S^{BB}\Gamma_D^B)^{-1}\,S^{BA}\bar{a} \tag{110}$$

Equations (102) and (110) describe the fields in region C.

Let S'_{BA} be defined as the vector $(T_{BA}, B_1, B_2, \ldots)$. Then

$$S'_{BA} = \Phi_D^B\,(I - S^{BB}\Gamma_D^B)^{-1}\,S^{BA}\bar{a} \tag{111}$$

Equations (103) and (111) describe the fields in region B.

Note, however, that the order of the matrix $(I - S^{BB}\Gamma_D^B)$ is infinite, and no method is now known to invert this matrix exactly. It will be shown that accurate calculations of such desirable quantities as the reflection co-

efficient R_A, using Equation (109), are possible by working with finite order matrices, i.e., by including in the calculation a finite number of elements in the matrix $(I - S^{BB}\Gamma_D^B)$. The rapid convergence of the solution with an increase in the truncation size is demonstrated in the following discussion.

Consider Equation (109). Let the matrix $(I - S^{BB}\Gamma_D^B)$ be truncated to a matrix of finite order N, $N > 1$. This means that only the first N rows and columns of the truncated matrix $(I - S^{BB}\Gamma_D^B)$ are included in the calculation. Let the determinant of the truncated matrix be denoted by $\Delta_{(N)}$. Let the determinant of the minor of the truncated matrix obtained by striking out the first row and column be denoted by $\Delta_{11\ (N-1)}$. Then, it can be shown that R_A is given approximately by

$$R_A \simeq \frac{b}{a}\,\rho_o^B\,\frac{\Delta_{11\ (N-1)}}{\Delta_{(N)}} \tag{112}$$

Of course, if $N = 1$, then Equation (109) reduces to a scalar equation for R_A which can be written as

$$R_A = \frac{b}{a}\,\frac{\rho_o^B}{1 - S_{oo}^{BB}\,\rho_o^B} \tag{113}$$

where S_{oo}^{BB} is given by

$$S_{oo}^{BB} = \frac{c}{a} \Big/ \left[-\frac{2ka_L}{\pi} - 2\sum_{n=1}^{\infty} \left\{ \tan^{-1}\left|\frac{k}{\alpha_n}\right| - \tan^{-1}\left|\frac{k}{\beta_n}\right| - \tan^{-1}\left|\frac{k}{\gamma_n}\right| \right\} \right], \quad 0 \le a/\lambda \le 0.5 \tag{114}$$

ρ_o^B as defined above is given by

$$\rho_o^B = \frac{1-\sqrt{K}}{1+\sqrt{K}} \tag{115}$$

It was demonstrated numerically that the approximate expression for R_A given by Equation (112) rapidly converges to a limit as the order N of the truncated matrix is increased. The results for a particular example are shown in Table 3 below. The choice of parameters for this set of calculations are $\kappa = 2.5$, $a/\lambda = .339$, and $c/a = .5$. Polystyrene has a relative dielectric constant around 2.5 and is a commonly used dielectric. The actual values of the elements comprising the first five rows and columns of the matrix $(I - S^{BB}\Gamma_D^B)$ for this set of parameters used in this calculation are shown in Table 4.

The results of the calculations indicate that the major contribution to the value of the reflection coefficient R_A comes from the term given by Equation (113), i.e., the term due to the TEM mode alone. The contribution to the value of R_A due to the higher order TM_{ro} modes is small. Indeed, for $a/\lambda << .5$, Equation (113) is an accurate expression for R_A.

TABLE 3

Reflection Coefficient R_A for Inhomogeneous E-Plane Bifurcation ($\kappa = 2.5$, $a/\lambda = .339$, $c/a = .5$)

Rank N of the Truncated Matrix $(I - S^{BB}\Gamma_D^B)$	Reflection Coefficient R_A
1	$-.105e^{j5.1^\circ}$
2	$-.107e^{j5.4^\circ}$
3	$-.107e^{j5.4^\circ}$
4	$-.107e^{j5.4^\circ}$

A system of infinite order linear algebraic equations can be derived for the inhomogeneous bifurcation. The derivation of this system of equations, given by Mittra and Pace[16], is essentially the same as the derivation of the

TABLE 4

ELEMENTS OF FIRST FIVE ROWS AND COLUMNS OF $(I - S^{BB}\Gamma_D^B)$

$(\kappa = 2.5,\ a/\lambda = .339,\ c/a = .5)$

$(I - S^{BB}\Gamma_D^B) =$

N = 1	N = 2	N = 3	N = 4	
.9436 + j.0973	.1133 - j.0668	-.0762 + j.0447	.0626 - j.0368	-.05
.0226 + j.0386	.9685 - j.0260	.0307 + j.0174	-.0289 - j.0143	.02
-.0077 - j.0131	.0157 + j.0089	.9833 - j.0059	.0167 + j.0049	-.01
.0042 + j.0071	-.0097 - j.0048	.0111 + j.0032	.9885 - j.0026	.01
-.0027 - j.0046	.0068 + j.0031	-.0080 - j.0021	.0086 + j.0017	.99

TABLE 4

IVE ROWS AND COLUMNS OF $(I - S^{BB}\Gamma_D^B)$

5, $a/\lambda = .339$, $c/a = .5$)

N = 2		N = 3		N = 4
- j.0668	-.0762 + j.0447	.0626 - j.0368	-.0538 + j.0316	
- j.0260	.0307 + j.0174	-.0289 - j.0143	.0268 + j.0123	
+ j.0089	.9833 - j.0059	.0167 + j.0049	-.0161 - j.0042	
- j.0048	.0111 + j.0032	.9885 - j.0026	.0114 + j.0023	
+ j.0031	-.0080 - j.0021	.0086 + j.0017	.9914 - j.0015	

system of equations corresponding to the homogeneous bifurcation (see Section 3.1). However, unlike the equations derived in Section 3.1, the set of equations for the inhomogeneous bifurcation cannot be solved exactly. An iteration technique must be used for its solution. A relationship has been shown between the iterative solution of the system of equations pertaining to the inhomogeneous bifurcation and the scattering matrix formulation of the same problem. The iterative solution of the above-mentioned system of equations is not nearly as convenient for purposes of calculation as say Equation (109).

Cronson[28] in the only known paper that deals specifically with the problem of the inhomogeneous E-plane bifurcation has derived an equivalent circuit for the junction. Refer to Figure 10. Cronson expresses the equivalent junction capacitance C_A in terms of an infinite series of sine terms with constant coefficients. The coefficients are the solutions to a system of infinite order linear algebraic equations. Cronson finds that it is not possible to solve his system of equations exactly. A method of approximation is employed. He resorts to solving a truncated set of equations. Specifically, he solves a system of equations of rank 6. The use of a digital computer is required to carry out all of the computations, including the computations of the elements of the sixth order matrix to be inverted.

Cronson checks the validity of his approximations by comparing his results for the case $\kappa = 1$ with the results given by Marcuvitz in the "Waveguide Handbook"[17] for the homogeneous bifurcation. Working with the normalized capacitance $C_V (C_V = - C_A/\omega\epsilon_o)$, Cronson finds the following percent errors in his calculations: $c/a = .5$, $a/\lambda = .5$, 2.29%; $c/a = .5$, $a/\lambda = .3$, 3.46%; $c/a = .5$, $a/\lambda = .5$, 8.52%.

The percentage error in computing R_A from C_V is less than the percentage error in C_V itself. For instance, for the set of parameters $a/\lambda = .339$,

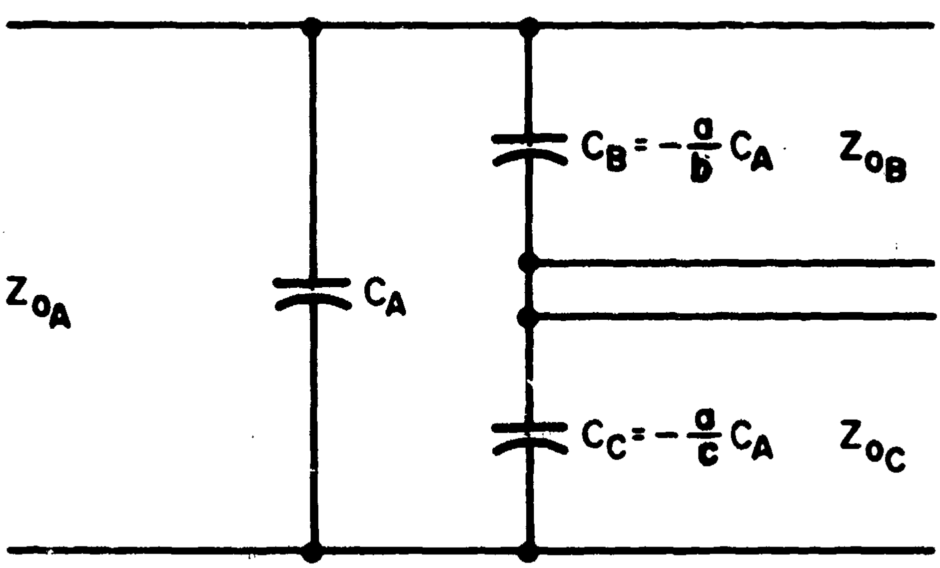


Figure 10. Equivalent circuit for inhomogeneous E-plane bifurcation.*

*Z_{oA}, Z_{oB}, and Z_{oC} are the characteristic impedances of Regions A, B, and C, respectively.

$c/a = .5$ and $\kappa = 2.5$, Cronson computed $C_V = .31$. From this value for C_V, one calculates $R_A = -.108e^{j5.5^\circ}$. This is to be compared with the results shown in Table 3. Using a matrix $(I - S^{BB}\Gamma_D^B)$ of fourth order, $R_A = -.107e^{j5.4^\circ}$ was calculated.

5. THE E-PLANE METALLIC STEP DISCONTINUITY

Much has been written in recent years about the problem of the step discontinuity in a waveguide. Refer to Figure 1 . The inclusion here of a discussion of the step discontinuity problem is justified on two counts. First, from the viewpoint of studying the generalized scattering matrix formulation, the step discontinuity is particularly interesting because it represents a 'worst possible case'. The configuration of the auxiliary problem is modified by placing a perfectly conducting wall in region B flush with the plane of the junction, which implies $\Gamma^B = -I$. It is expected that the effect of the higher order modes is greater in this case than in the preceding problem of the inhomogeneous bifurcation. It is desirable to show that even in an extreme case, important quantities such as the reflection coefficient R_A can be computed easily because of the rapid convergence of the matrix series expansion.

The second point is that the step discontinuity problem warrants attention for its own sake. It is a classic problem, studied by a number of authors using more established methods. Macfarlane[29] and Marcuvitz[17] have found quasi-static solutions to the step discontinuity problem. Their methods differ in the exact details, but essentially are the same. The equivalent susceptance of the waveguide junction is formulated in terms of an integral equation, the exact solution of which is not possible in general. The equation is solvable for the case of $k\ (2\pi/\lambda) = 0$. The static field problem is solved by simplifying the original problem through conformal transformations. Extensive results are tabulated in the "Waveguide Handbook". Only the case of single mode propagation is considered. In this frequency range the 'equivalent static' method yields very accurate answers. However, the method becomes quite involved for the case when the dimensions of the guide are such that several

modes propagate, i.e., a multi-mode waveguide. The multi-mode problem is assuming increased importance. It has applications in the field of millimeter wave propagation[30] and the study of VLF propagation[31].

Schwinger[32] has solved the above-mentioned integral equation for the equivalent susceptance by means of a variational technique. However, the accuracy of the variational technique is dependent upon the choice of the trial function and, in this sense, is not a deterministic method. Furthermore, the choice of the trial function in the variational technique is not at all straightforward for multi-mode propagation.

As previously discussed in Section 2.3, Williams[22] has applied the Wiener-Hopf technique to the step discontinuity problem. Williams' method was outlined in Section 2.3 and the differences between his approach to the step discontinuity problem and the approach based on the generalized scattering matrix technique were discussed. Williams also includes some numerical results in his paper. For the case of $a/\lambda < .5$, his numerical results are in close agreement with Marcuvitz's results. Williams also discusses the situation when two modes are allowed to propagate in the larger channel of the guide (region A), including numerical values of the square of the magnitude of the reflection coefficient R_A. Williams' results will be used for comparison with the results reported in the section.

There is still active interest in the step discontinuity problem as evidenced by the most recent paper on the subject by Magnus and Fox[33]. In this paper, the problem is treated as an infinite set of inhomogeneous linear equations. They are solved formally by a perturbation technique.

With this brief introduction, now consider the problem of the step discontinuity in terms of the generalized scattering matrix formulation. With

reference to Figure 1, assume that a TEM mode of unit amplitude is incident in region A, traveling in the positive z direction. As in Section 4, the non-zero field components H_y, E_x, and E_z can be derived from a scalar function $\psi(x, z)$. In this case, ψ satisfies the homogeneous Helmholtz equation together with the boundary conditions

$$\frac{\partial\psi}{\partial x} = 0, \; x = 0, a, \text{ all } z \text{ and } x = c, \; z > 0. \tag{116}$$

and

$$\frac{\partial\psi}{\partial z} = 0, \; z = 0, \; c \leq x \leq a \tag{117}$$

ψ also satisfies the edge condition given by

$$\left|\nabla \psi\right| = O(d^{-1/3}), \; d \longrightarrow 0 \tag{118}$$

where $d = [(x - c)^2 + z^2]^{1/2}$

Let the fields in region A be expanded in the cosine series given by Equation (101). The mode coefficients of the reflected field are given by

$$\bar{S}'_{in} = - S^{AB} (I + S^{BB})^{-1} S^{BA} \bar{a} \tag{119}$$

where $\bar{S}_{in}$ is the column vector defined in Section 4. Here, $\Gamma^B = -I$ where I is the identity matrix.

Similarly, let the fields in region C be expanded in the cosine series given by Equation (101). Then the mode coefficients expressed by the column vector $\bar{S}_{CA}$ are given by

$$\bar{S}'_{CA} = S^{CA} \bar{a} - S^{CB} (I + S^{BB})^{-1} S^{BA} \bar{a} \tag{120}$$

The fields in region B, of course, are identically zero.

That the solution expressed by Equations (119) and (120) satisfies the edge condition given by Equation (118) has not been shown. In order to do this, one should examine the asymptotic behaviour of the higher order mode coefficients, i.e., the higher order elements of the vectors $\overline{S}'_{in}$ or $\overline{S}_{CA}$. In this section, only the reflection coefficient R_A is computed. However, the proof of the convergence of the Neumann series $I + S^{BB} + S^{BB}S^{BB} + \ldots$ is given in Section 2.2. Also, it is demonstrated in this section that for $0 \leq a/\lambda \leq 1.0$, the computed values for R_A are in close agreement with figures computed from the expression given in the "Waveguide Handbook", or alternately, with figures reported by Williams.[22]

Now in Equation (119), let the matrix $(I + S^{BB})$ be truncated to a matrix of order N, $N > 1$. As in Section 4, let the determinant of the truncated matrix be denoted by $\Delta_{(N)}$ and let the determinant of the minor of the truncated matrix obtained by striking out the first row and column be denoted by $\Delta_{11(N-1)}$. Then R_A is given approximately by

$$R_A \simeq - \frac{b}{a} \frac{\Delta_{11(N-1)}}{\Delta_{(N)}} \tag{121}$$

If $N = 1$, then

$$R_A \simeq - \frac{b}{a} \frac{1}{1 + S^{BB}_{oo}} \tag{122}$$

where S^{BB}_{oo} is given by Equation (114) for $0 \leq a/\lambda \leq 0.5$.

Even in this case, it was demonstrated numerically that the approximate Equation (121) for R_A converges rapidly to a limit as the order N of the

truncated matrix $(I + S^{BB})$ increases. The results of two sets of calculations are shown in Tables 5 and 6. The results of the calculations indicate that in the range $0 \leq a/\lambda \leq .5$, a truncated matrix $(I + S^{BB})$ of order four or five is sufficient for purposes of accurate computation.

For $c/a = .5$ and $a/\lambda = .339$, $R_A = -.393e^{j20.6^\circ}$ was calculated from the expression for the equivalent susceptance of the junction given by Marcuvitz in the "Waveguide Handbook". Compare this with the value $R_A = .392e^{j21.3^\circ}$ calculated from Equation (121) with $N = 5$. The actual values of the elements comprising the first five rows and columns of the matrix $(I + S^{BB})$ for this set of parameters used in the calculations are shown in Table 7. In the second example with $c/a = .326$ and $a/\lambda = .3$, $R_A = -.215e^{j20.9^\circ}$ was computed from Marcuvitz's figures. $R_A = -.215e^{j20.3^\circ}$ was calculated from Equation (121) with $N = 4$.

TABLE 5

Reflection Coefficient R_A for Step Discontinuity

$(a/\lambda = .339,\ c/a = .5)$

Rank N of Truncated Matrix $(I + S^{BB})$	Reflection Coefficient R_A
1	$-.379e^{j18.8^\circ}$
2	$-.388e^{j20.7^\circ}$
3	$-.391e^{j21.1^\circ}$
4	$-.392e^{j21.2^\circ}$
5	$-.392e^{j21.3^\circ}$

TABLE 6

Reflection Coefficent R_A for Step Discontinuity

($a/\lambda = .3$, $c/a = .326$)

Rank of Truncated Matrix $(I + S^{BB})$	Reflection Coefficient R_A
1	$-.210e^{j19.6^{\circ}}$
2	$-.214e^{j19.8^{\circ}}$
3	$-.215e^{j20.0^{\circ}}$
4	$-.215e^{j20.3^{\circ}}$

Next, consider the situation when the TEM and TM_{10} modes are allowed to propagate in region A, but only the TEM mode propagates in region C. The reflection coefficient R_A was computed using Equation (121) and Equation (122). The results of the numerical calculations for two examples are listed in Tables 8 and 9. One can compare these results with those published by Williams. For $a/\lambda = 0.7$ and $c/a = 0.5$, Williams computes $\left|R_A\right|^2 = .28$. Using Equation (121) with $N = 5$, one computes $\left|R_A\right|^2 = .29$. The values of the elements of the first five rows and columns of $(I + S^{BB})$ for this set of parameters are shown in Table 10. For $a/\lambda = 0.9$ and $c/a = 0.5$, Williams computes $\left|R_A\right|^2 = .25$. Again using Equation (121), $N = 5$, one can compute $\left|R_A\right|^2 = .25$. Thus, there is close agreement between the results reported by the author and Williams' results even when more than one mode is allowed to propagate in region A. The dominant contribution to R_A comes from Equation (122) with S_{oo}^{BB} given by

$$S_{oo}^{BB} = \frac{c}{a}\,\frac{\left|\alpha_1\right| - k}{\left|\alpha_1\right| + k}\;\Big/ - \frac{2ka_L}{\pi} - 2\sum_{n=2}^{\infty} \tan^{-1}\left|\frac{k}{\alpha_n}\right| + 2\sum_{n=1}^{\infty}\left\{\tan^{-1}\left|\frac{k}{\gamma_n}\right| + \tan^{-1}\left|\frac{k}{\beta_n}\right|\right\} \tag{123}$$

TABLE 7

ELEMENTS OF FIRST FIVE ROWS AND COLUMNS OF $(I + S^{BB})$

$(a/\lambda = .339,\ c/a = .5)$

$(I + S^{BB}) =$

N = 1	N = 2	N = 3	N = 4	
1.2508 - j.4326	-.2411 + j.1416	.1736 - j.1019	-.1445 + j.0849	.1
-.1006 - j.1714	1.0667 + j.0551	-.0698 - j.0396	.0667 - j.0329	-.0
.0342 + j.0583	-.0332 - j.0188	1.038 + j.0135	-.0386 - j.0112	.0
-.0186 - j.0316	.0206 + j.0102	-.0252 - j.0073	1.0265 + j.0061	-.0
.0121 + j.0205	-.0144 - j.0066	.0183 + j.0048	-.0198 - j.0040	1.0

TABLE 7

OF FIRST FIVE ROWS AND COLUMNS OF $(I + S^{BB})$

$(a/\lambda = .339,\ c/a = .5)$

= 1	N = 2	N = 3	N = 4
-.2411 + j.1416	.1736 - j.1019	-.1445 + j.0849	.1249 - j.0733
1.0667 + j.0551	-.0698 - j.0396	.0667 - j.0329	-.0622 - j.0286
-.0332 - j.0188	1.038 + j.0135	-.0386 - j.0112	.0373 + j.0097
.0206 + j.0102	-.0252 - j.0073	1.0265 + j.0061	-.0263 - j.0053
-.0144 - j.0066	.0183 + j.0048	-.0198 - j.0040	1.0200 + j.0034

TABLE 8

REFLECTION COEFFICIENT R_A FOR STEP DISCONTINUITY

TWO MODES PROPAGATING IN REGION A

($a/\lambda = 0.7$, $c/a = 0.5$)

Rank N of Truncated Matrix $(I + S^{BB})$	Reflection Coefficient R_A
1	$-.52e^{j4.6^\circ}$
2	$-.52e^{j2.8^\circ}$
3	$-.52e^{j2.2^\circ}$
4	$-.52e^{j2.2^\circ}$
5	$-.52e^{j2.2^\circ}$

TABLE 9

REFLECTION COEFFICIENT R_A FOR STEP DISCONTINUITY

TWO MODES PROPAGATING IN REGION A

($a/\lambda = 0.9$, $c/a = 0.5$)

Rank N of Truncated Matrix $(I + S^{BB})$	Reflection Coefficient R_A
1	$-.50e^{j2.6^\circ}$
2	$-.50e^{j3.7^\circ}$
3	$-.50e^{j1.7^\circ}$
4	$-.50e^{j1.7^\circ}$
5	$-.50e^{j1.7^\circ}$

TABLE 10

ELEMENTS OF FIRST FIVE ROWS AND COLUMNS OF $(I + S^{BB})$

TWO MODES PROPAGATING IN REGION A

$(a/\lambda = 0.7,\ c/a = .5)$

$(I + S^{BB}) =$

	N = 1	N = 2	N = 3	N = 4	
	.9553 - j.0762	.0164 + j.1499	.0218 - j.0906	-.0236 + j.0702	.
	-.2849 + j.0312	1.2251 + j.0817	-.1591 - j.0450	.1359 + j.0337	-.
	.0671 + j.0161	-.0615 - j.0174	1.0539 + j.0087	-.0505 - j.0087	.
	-.0332 - j.0112	.0337 + j.0083	-.0323 - j.0056	1.0317 + j.0044	-.
	.0206 + j.0079	-.0222 - j.0051	.0225 + j.0034	-.0228 - j.0028	1.

TABLE 10

TS OF FIRST FIVE ROWS AND COLUMNS OF $(I + S^{BB})$

TWO MODES PROPAGATING IN REGION A

$(a/\lambda = 0.7,\ c/a = .5)$

	N = 1	N = 2	N = 3	N = 4
2	.0164 + j.1499	.0218 - j.0906	-.0236 + j.0702	.0228 - j.0593
2	1.2251 + j.0817	-.1591 - j.0450	.1359 + j.0337	-.1215 - j.0280
1	-.0615 - j.0174	1.0539 + j.0087	-.0505 - j.0087	.0476 + j.0074
2	.0337 + j.0083	-.0323 - j.0056	1.0317 + j.0044	-.0309 - j.0038
9	-.0222 - j.0051	.0225 + j.0034	-.0228 - j.0028	1.0226 + j.0024

6. THE TRIFURCATED WAVEGUIDE

This section is concerned with the boundary value problem associated with two semi-infinite plates in a parallel plate waveguide. Refer to Figure 3. Let a TEM mode be incident in region C. Let ψ (x, z) be a scalar function such that $\psi = H_y$. Then, E_x and E_y can be derived from ψ using Equations (93) and (94) with $\epsilon = \epsilon_o$. ψ satisfies the homogeneous Helmholtz equation together with the boundary conditions

$$\frac{\partial\psi}{\partial x} = 0, \; x = 0, \; a \text{ for all } z \text{ and } x = c, \; h \quad \text{for } z > 0 \tag{124}$$

and the edge conditions

$$|\nabla\psi| = O\left(d_{1,2}^{-1/2}\right), \; d_{1,2} \longrightarrow 0 \tag{125}$$

where $d_1 = [(x - c)^2 + z^2]^{1/2}$

and $d_2 = [(x - h)^2 + z^2]^{1/2}$

$\psi(x,z)$ can be expanded in region C in the cosine series given by

$$\psi_C = e^{-j\gamma_o z} - R_C e^{j\gamma_o z} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right) e^{j\gamma_n z} \tag{126}$$

where R_C is the voltage reflectance coefficient for the TEM mode. The amplitude of the incident TEM mode in region C is assumed to be one.

An expression for the coefficients of the Fourier series expansion given by Equation (126) can be written in terms of the scattering coefficients of the auxiliary problem and the load in region B. Thus, one can show that

$$\bar{S}''_{in} = S^{CC}\,\bar{c} + S^{CB}\Gamma^{B}\left(I - S^{BB}\Gamma^{B}\right)^{-1} S^{BC}\,\bar{c} \tag{127}$$

where $\bar{S}''_{in} = (-R_C, C_1, C_2, \ldots)^T$ and $\bar{c} = (1, 0, 0, \ldots)^T$. The elements of Γ^B in this case are known. As mentioned before, the bifurcated waveguide is modified by placing a second semi-infinite plate in region B. Thus, the derivation of the elements of Γ^B for this problem is the same as the one followed in solving for the elements of S^{AA} in Section 2. To derive the elements of Γ^B from the expressions for S^{AA}_{mn} given in Table 2, simply replace a by b, b by h - c, and b by a - h.

Note the semi-infinite plate in region B is coupled electro-magnetically to region C by the higher order TM_{no} modes. A TEM mode scattered in region B, traveling in the positive z direction, will not be reflected by the septum in that region. The higher order TM_{no} will be reflected, however, and will contribute some to the final value of the reflection coefficient R_C. The dominant term in the series expansion for R_C is given by S^{CC}_{oo} since as shown by actual calculation, the contribution of the higher order TM_{no} modes is numerically small compared to S^{CC}_{oo}. This is to say that the second plate can be introduced in region B without appreciably affecting the reflection coefficient R_C.

The reflection coefficient R_C can be computed from Equation (127) if first each of the matrices appearing in the equation are replaced by a matrix of order N. It was demonstrated by means of actual calculations that in the case of both the step discontinuity and the inhomogeneous bifurcation, the expression for the reflection coefficient rapidly converges to a limit as the order N of the truncated matrices increases. As examples, the numerical results of two sets of calculations are cited in Tables 11 and 12. The parameters used in the first example are $h/a = .5$, $c/h = .5$, and $a/\lambda = .4$. In the second example, they are $h/a = .326$, $c/h = .326$, and $a/\lambda = .3$. In both of these examples, the two plates are asymmetrically situated with respect to the center

TABLE 11

REFLECTION COEFFICIENT R_C FOR TRIFURCATED WAVEGUIDE

($h/a = .5$, $c/h = .5$, $a/\lambda = .4$)

No. of Modes (N) Included in Calculations	Reflection Coefficient R_C
1	$.500e^{-j75.5^\circ}$
2	$.470e^{j78.4^\circ}$
3	$.470e^{-j78.9^\circ}$
4	$.475e^{-j77.9^\circ}$
6	$.474e^{-j78.1^\circ}$

TABLE 12

REFLECTION COEFFICIENT R_C FOR TRIFURCATED WAVEGUIDE

($h/a = .326$, $c/h = .326$, $a/\lambda = .3$)

No. of Modes (N) Included in Calculations	Reflection Coefficient R_C
1	$.326e^{-j46.9^\circ}$
2	$.327e^{-j48.0^\circ}$
3	$.324e^{-j47.5^\circ}$
4	$.324e^{-j47.5^\circ}$
6	$.324e^{-j47.4^\circ}$

line of the waveguide described by x = a/2. It is shown below that a simple expression can be derived for R_C if the two plates are symmetrically placed with respect to x = a/2.

It is possible to formulate the problem of the trifurcated waveguide in terms of a system of Wiener-Hopf integral equations. A. E. Heins[34] discusses the special case of an arbitrary number of equally spaced semi-infinite plates in a waveguide. The set of integral equations can be formulated in terms of the unknown current densities on each of the semi-infinite plates. For the case of the trifurcated waveguide, the system of integral equations are of the form

$$\sum_{j=1}^{2} \int_{o}^{\infty} K_{ij} (z - z') J_i (z') dz' + F_j(z) = 0 \qquad (128)$$

for z > 0 and i = 1, 2. $K_{ij} (z - z')$ are linear combinations of the Green's functions used in formulating the integral equations and $F_j(z)$ is the form of the propagating modes in the j^{th} duct. The solution of Equation (128) is complicated. To solve the above set of equations, the Wiener-Hopf technique must be generalized. It is necessary to factorize the determinant of the matrix whose elements are the Fourier transforms of the kernels $K_{ij}(z)$. In this instance, one must factorize a determinant of order two. Heins has discussed this problem in general terms, but he has not, as far as it is known, published the actual solution to the problem. Wu and Wu[35] in a paper published much later than Heins state that in the case of coupled Wiener-Hopf integral equations, solutions are not known except for the cases where reduction to a single equation is possible.

The exact solution to the trifurcated waveguide is possible when the plates are symmetrically spaced with respect to the center line x = a/2 as

mentioned above. This is discussed, next.

It is desired to find the reflection coefficient R_C for the dominant mode incident in region C. Refer again to Figure 3. This mode of excitation can be broken into two separate cases of even and odd excitation. Consider Figure 11. The area of the three smaller ducts to the right of the plane of the junction are labeled B', B'', and C. In the case of even excitation (Figure 11a), regions B' and C are excited in the TEM mode so that the incident fields are in phase. In the case of odd excitation (Figure 11b), regions B' and C are excited in the TEM mode so that the incident mode in region B' is out of phase with the incident mode in region C. Each of these problems can be solved individually for the reflection coefficient of the TEM mode in region C. Let R_e be the voltage reflection coefficient of the TEM mode in region C for the case of even excitation. Similarly, let R_o be the voltage reflection coefficient for the TEM mode in region C for the case of odd excitation. Then by super-position,

$$R_C = \frac{1}{2} (R_e + R_o) \tag{129}$$

R_e and R_o can be found quite simply. Consider first the case of even excitation. Refer to Figure 12a. Only symmetric transverse magnetic modes will be excited in region A, i.e., the TM_{no} modes where n = 0, 2, 4, ... Because of the symmetry involved, it is possible to place an electric wall at x = a/2 and solve the boundary value problem associated with the reduced geometry. This problem, of course, was solved in Section 3. As a matter of

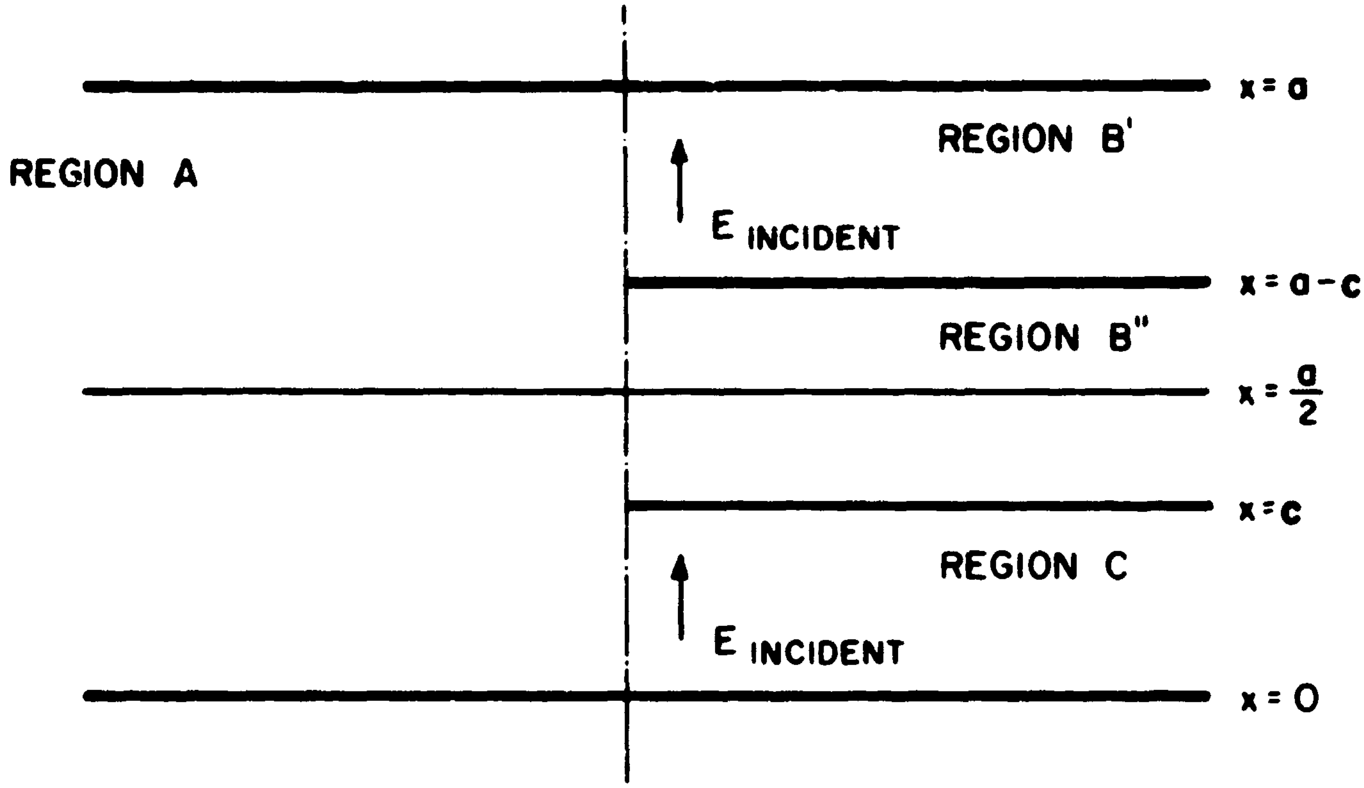


Figure 11a. Even mode of excitation.

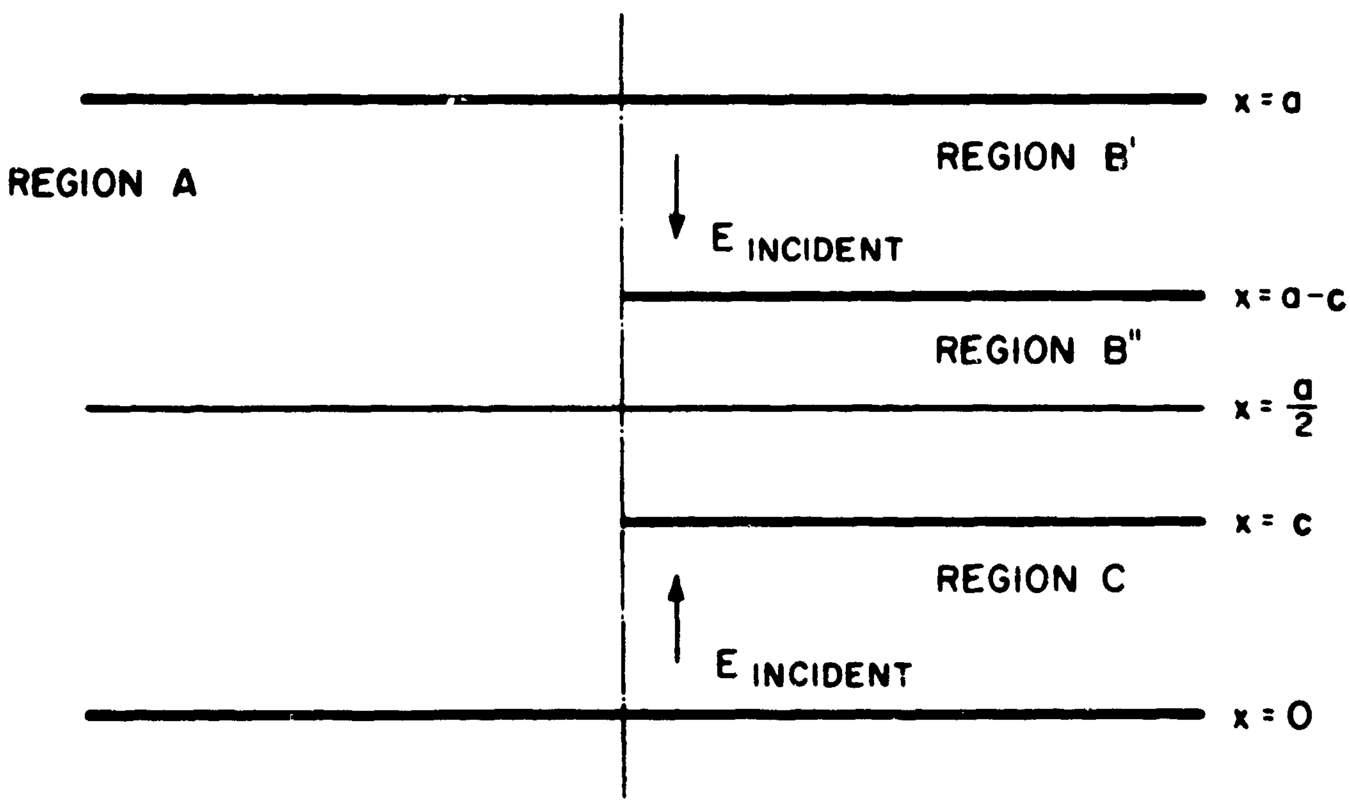
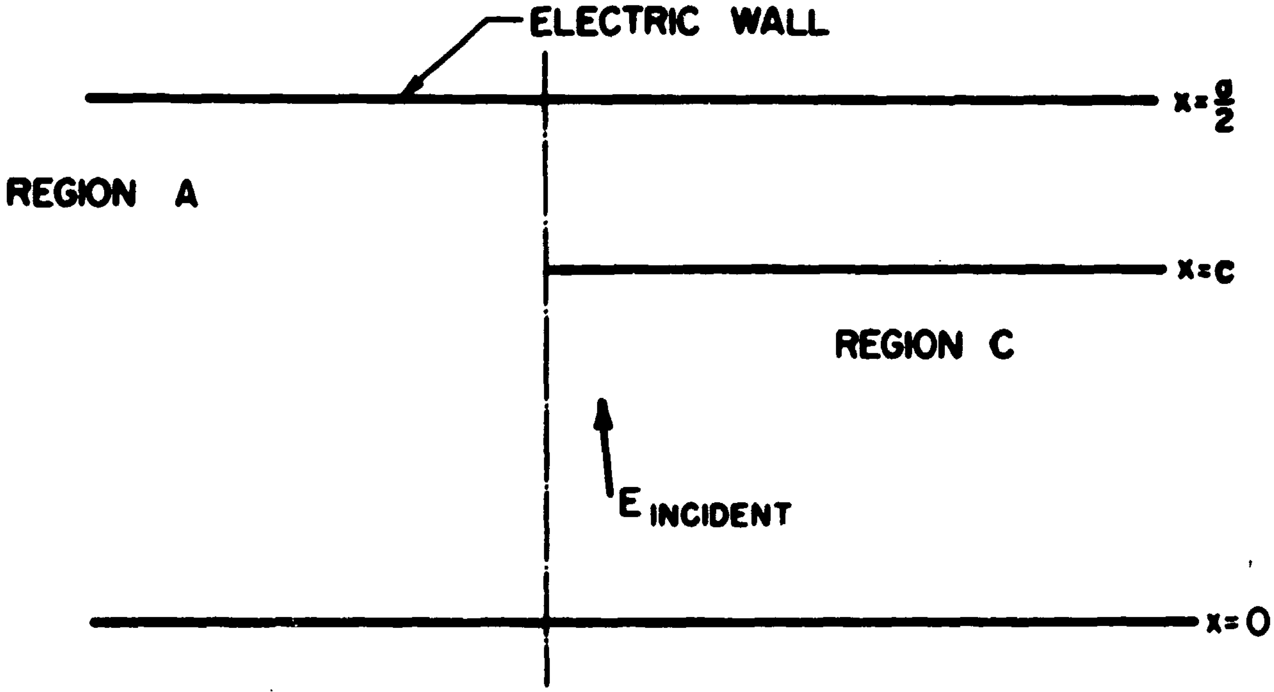
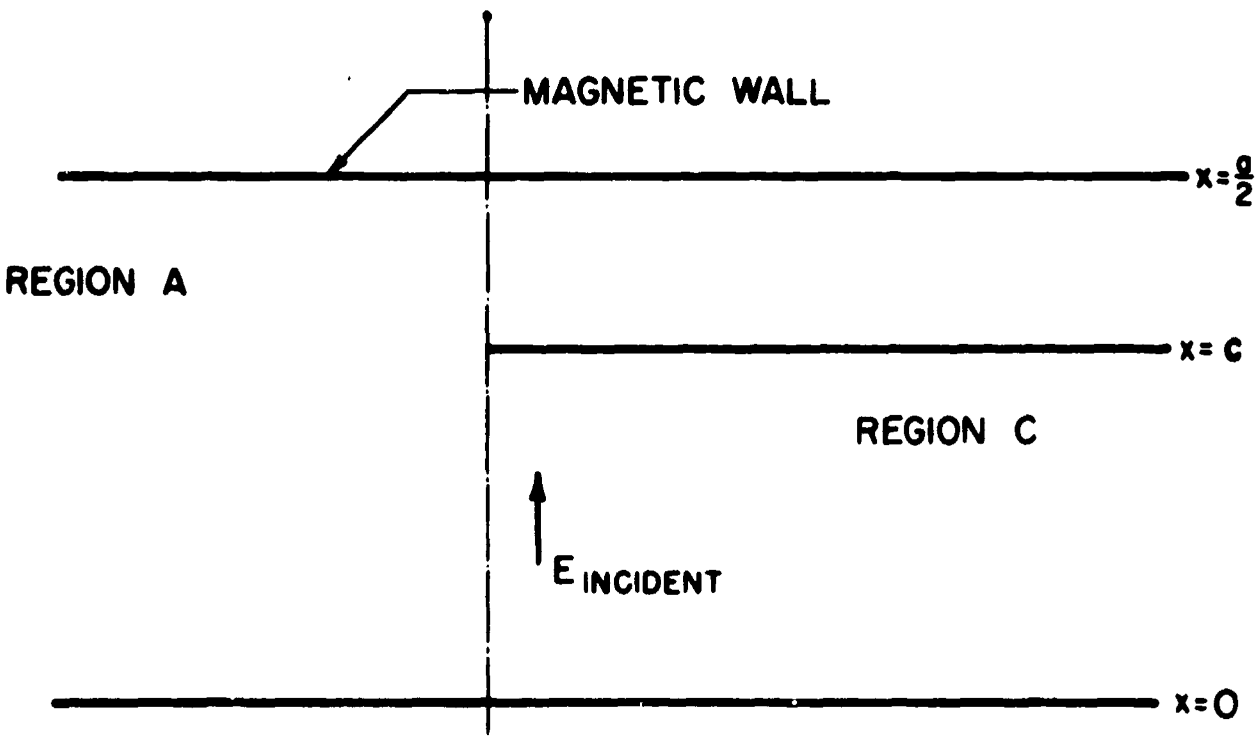


Figure 11b. Odd mode of excitation.



(a)

Figure 12a. Problem associated with even excitation.



(b)

Figure 12b. Problem associated with odd excitation.

convenience, one can define the propagation constant ζ_n for region B by

$$\zeta_n = \sqrt{k^2 - \left(\frac{\pi n}{a - 2c}\right)^2}, \qquad k > \frac{\pi n}{a - 2c}$$

$$= -j\sqrt{\left(\frac{\pi n}{a - 2c}\right)^2 - k^2}, \qquad \frac{\pi n}{a - 2c} > k$$

Then, after making the appropriate substitutions in the known expressions for S_{oo}^{CC} one can write

$$R_e = \frac{a - 2c}{a} e^{jX_e} \tag{130}$$

where

$$X^e = -\frac{ka}{\pi}\left\{\ell n\left(\frac{a}{a - 2c}\right) + \frac{2c}{a}\ell n\left(\frac{a - 2c}{2c}\right)\right\}$$
$$+ 2\sum_{n=0}^{\infty}\left\{\tan^{-1}\left|\frac{k}{\zeta_{2n}}\right| + \tan^{-1}\left|\frac{k}{v_n}\right| - \tan^{-1}\left|\frac{k}{\lambda_{2n}}\right|\right\} \tag{131}$$

Consider next the odd mode of excitation. Refer to Figure 12b. Only asymmetric transverse magnetic modes will be excited in region A, i.e., the TM_{no} modes where n = 1, 3, 5, ... A simpler but equivalent boundary value problem is obtained by placing a magnetic wall at x = a/2. The solution of this problem is quite straight forward. In the manner illustrated in Section 3, a system of infinite order linear algebraic equations are derived which can be solved by means of the function-theoretic technique. The solution of this problem yields

$$R_o = e^{jX_o} \tag{132}$$

where

$$X_o = -\frac{ka}{\pi}\left\{\ln\left(\frac{2a}{c}\right) + \left(\frac{a-2c}{a}\right)\ln\left(\frac{c}{2(a-2c)}\right)\right\}$$
$$+ 2\sum_{n=1}^{\infty}\left\{\tan^{-1}\left|\frac{k}{\alpha_n}\right| + \tan^{-1}\left|\frac{k}{\zeta_{2n}}\right| - \tan^{-1}\left|\frac{k}{\gamma_n}\right|\right.$$
$$\left. - \tan^{-1}\left|\frac{k}{\zeta_{3n}}\right| - \tan^{-1}\left|\frac{k}{\alpha_{2n}}\right|\right\} \qquad (133)$$

As an example of the application of Equation (129), consider the problem of two equally spaced semi-infinite plates situation in a waveguide. Let $a/\lambda = .3$. Using Equation (129), one computes $R_C = .646\ e^{-j49.0^\circ}$. For purposes of comparison, R_C was also computed from Equation (127). With $N = 4$, $R_C = .643e^{-j49.0^\circ}$ was calculated, which is in very close agreement.

It should be stressed again that Equation (129) is valid only for the special case of symmetrically spaced plates. No simple expression can be found for the more general problem of arbitrarily spaced plates.

7. CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

The generalized scattering matrix technique has been introduced and applied to three waveguide discontinuity problems. They are the E-plane metallic step discontinuity, the inhomogeneous E-plane bifurcation, and the trifurcated waveguide. The solutions obtained are formally exact, though in series form.

To the best of the author's knowledge, the general trifurcation problem has not been discussed elsewhere. The solutions to the other two problems, viz., the metallic step discontinuity and inhomogeneous bifurcated guide have been derived, at least approximately, by other authors using conventional methods of analysis.

Numerical calculations have been carried out for each of the three problems described above. The results of the numerical calculations help to demonstrate the utility and potential accuracy of the generalized scattering matrix technique applied to waveguide discontinuity problems. It is shown that the series expression for the reflection coefficient of the dominant mode in the frequency range $0 \leq a/\lambda \leq 1.0$ is rapidly convergent, thereby making calculations convenient. Moreover, the numerical results, where possible, are compared with published results of other authors and they are found to be in close agreement with one another.

The successful application of the generalized scattering matrix technique to waveguide discontinuity problems is a first step in showing its applicability to a broader range of problems. A list of suggested problems for future study together with the corresponding auxiliary problems is given below.

1. Dielectric step discontinuity in a waveguide.
2. The diffraction of a plane wave by a dielectric grating.

The suggested auxiliary problem for 1. and 2. is a semi-infinite

impedance wall bifurcating a waveguide. The impedance wall is characterized by an impedance matrix Z.

3. The diffraction of a plane wave by a thick half-plane.

4. The diffraction of a plane wave by a solid, circularly shaped, metallic bar.

   The auxiliary problem is a semi-infinite, tubular waveguide in free space.

5. The asymmetrical inductive and capacitive diaphragms in a waveguide as well as the corresponding strip grating problems.

   The suggested auxiliary problem is the bifurcated waveguide already discussed in this thesis. Refer to Figure 4.

6. The study of the electromagnetic properties of certain types of grating structures. These structures have applications as surface or leaky wave antennas.

   The auxiliary problems for this kind of problem is discussed in detail by Mittra and Pace.[19]

BIBLIOGRAPHY

1. P. M. Morse and H. Feshback, Methods of Theoretical Physics, part I, McGraw-Hill Book Co., Inc., New York, 1953.
2. B. Noble, Methods Based on the Wiener-Hopf Technique, Permagon Press, Inc., New York, 1958.
3. L. Brillouin, "Waveguides for Slow Waves", J. Appl. Phys., 19, pp. 1023-1041, November, 1948.
4. E. A. N. Whitehead, "The Theory of Parallel Plate Media for Microwave Lenses", Proc. IEE (London), 98, part III, pp. 133-140, January, 1951.
5. Z. S. Agronovich, V. A. Marchenko, and V. P. Shestopalov, "The Diffraction of Electromagnetic Waves from Plane Metallic Lattices", J. Tech. Phys., 7, 4, pp. 277-286, October, 1962.
6. A. I. Adonina and V. P. Shestopalov, "Diffraction of Electromagnetic Waves Obliquely Incident on a Plane Metallic Grating With a Dielectric Layer", J. Tech. Phys., 8, No. 6, pp. 479-486, December, 1963.
7. R. A. Hurd and H. Gruenberg, "H-plane Bifurcation of Rectangular Waveguides", Can. J. Phys., 32, pp. 694-701, November, 1954.
8. D. R. Hartree, Numerical Analysis, Oxford Press, 1952.
9. A. S. Householder, Principles of Numerical Analysis, McGraw-Hill Book Co., Inc., New York, 1953.
10. G. Goertzel and N. Tralli, Some Mathematical Methods of Physics, McGraw-Hill Book Co., Inc., New York, 1960.
11. L. Lewin, Advanced Theory of Waveguides, Iliffe and Sons, LTD, London, 1951.
12. R. E. Collin, Field Theory of Guided Waves, McGraw-Hill Book Co., Inc., New York, 1960.
13. R. N. Ghose, Microwave Circuit Theory and Analysis, McGraw-Hill Book Co., Inc., New York, 1963.
14. S. H. Durrani, "Techniques for Solving Waveguide Discontinuity Problems", University of New Mexico Engineering Experiment Station, Tech Report EE-71, March, 1962.
15. A. F. Harvey, Microwave Engineering, Academic Press, New York, 1963.
16. R. Mittra and J. R. Pace, "A New Technique for Solving a Class of Boundary Value Problems", Transactions IEEE, AP-11, No. 5, p. 617, Sept. 1963. Also, Tech. Rep. 72, Antenna Laboratory, University of Illinois, Sept. 1963.

17. N. Marcuvitz, Waveguide Handbook, MIT Rad. Lab. Series, 10, McGraw-Hill Book Co., Inc., New York, 1951.

18. R. Mittra, "The Finite Range Wiener-Hopf Integral Equation and a Boundary Value Problem in a Waveguide", Transactions IRE, AP-7, Special Supplement, S244-S254, December, 1959.

19. R. Mittra and J. R. Pace, "A New Technique for Solving a Class of Boundary Value Problems", Antenna Lab., University of Illinois, Tech. Report No. 72, on Contract AF33(657)-10474, Sept., 1963.

20. C. G. Montgomery, R. H. Dicke, and E. M. Purcell, Principles of Microwave Circuits, MIT Rad. Lab. Series, 8, McGraw-Hill Book Co., Inc., New York, 1948.

21. B. Friedman, Principles and Techniques of Applied Mathematics, John Wiley and Sons, Inc., New York, 1956.

22. W. E. Williams, "Step Discontinuities in Waveguides", Transactions IRE, AP-5, No. 2, pp. 191-198, April, 1957.

23. R. A. Hurd, "The Propagation of an Electromagnetic Wave Along an Infinite Corrugated Surface", Can. J. Phys., 32, pp. 727-734, December, 1954.

24. W. Magnus and F. Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea Publishing Co., p. 2, 1943.

25. R. Mittra, "Relative Convergence of the Solution of a Doubly Infinite Set of Equations", Journal of Research of the National Bureau of Standards, 67D, No. 2, pp. 245-254, March-April, 1963.

26. J. Meixner, "The Behavior of Electromagnetic Fields at Edges", N. Y. University Institute of Mathematical Sciences, Research Report EM-72, December, 1954.

27. Schelkunoff, private correspondence.

28. H. M. Cronson, "Equivalent Circuit Parameter for an Inhomogeneous Bifurcated Waveguide", Brown University, Scientific Report AF 4561/13, Contract AF 19 (604)-4561, Brown University, September, 1961.

29. G. G. MacFarlane, "Quasi-stationary Field Theory and Its Applications to Diaphragms and Junctions in Transmission Lines and Waveguides", Jour. IEE (London), 93, part III A, pp. 703-719, 1946.

30. J. W. E. Griemsmann, "Oversized Waveguides", Microwaves, pp. 20-31, December, 1963.

31. S. W. Maley and E. Baker, "Effects of Wall Perturbations in Multi-Mode Waveguides", Symposium on the Ionosphere Propagation of VLF Radio Waves, Boulder, Colorado, August, 1963.

32. J. Schwinger, "Discontinuities in Waveguides", MIT Lecture notes (edited by D. S. Saxon).

33. D. Fox and W. Magnus, "Perturbation Method in a Problem of Waveguide Theory", Journal of Research of the National Bureau of Standards, 67D, No. 2, pp. 189-198, March-April, 1963.

34. A. E. Heins, "Systems of Wiener-Hopf Integral Equations and Their Application to Some Boundary Value Problems in Electromagnetic Theory", Proceedings of Symposia in Applied Mathematics of the American Mathematical Society, 2, pp. 76-81, July 29-31, 1948.

35. T. T. Wu and T. T. Wu, "Iterative Solutions of Wiener-Hopf Integral Equations", Quarterly of Applied Mathematics, XX, No. 4, pp. 341-352, January, 1963.